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AEROFOILS OF MAXIMUM THICKNESS RATIO FOR A GIVEN MAXIMUM PRESSURE COEFFICIENT

By A. R. MANWELL (*University College, Southampton*)

[Received 3 October 1947. Revised 2 December 1947]

SUMMARY

A relation is found which determines the symmetrical aerofoil which in streaming incompressible or subsonic compressible flow has the greatest thickness ratio for a given maximum pressure coefficient. The solution in the incompressible case coincides with that of a free streamline problem due to Riabouchinsky. The Kármán–Tsien solution is derived in a very similar form. It is also shown how in principle one experiment with an electric potential apparatus would solve the problem for any pressure-density law.

The method of deriving the relation can be extended to various similar problems, including axially symmetric flow, and the solution possesses the further property of maximizing the area and other functions of the contour of the aerofoil.

1. Introduction

THE problem to be solved is that of finding the shape of the symmetrical aerofoil which in unlimited streaming flow has the following properties:

For a given maximum pressure (or velocity) peak on the boundary

(*a*) the maximum thickness to chord ratio,

(*b*) the area to $(\text{chord})^2$ ratio

are as large as possible.

Lockwood Taylor (ref. **1**) argued that the most efficient method of obtaining lift would be to make the suction, and so the velocity, constant over the boundary. Using thin aerofoil theory he calculated the approximate form of both cambered and symmetrical aerofoils having constant velocity portions. In an unpublished paper, L. G. Whitehead in 1942† has worked out a number of similar aerofoils of finite thickness having constant velocity arcs. He observes in his paper that a free streamline problem solved by Riabouchinsky (ref. **2**) has, of all other examples known to him, the best thickness ratio in the sense defined earlier in this paragraph. The present paper is, so far as the author is aware, the first to attempt to formulate and discuss the problem precisely. Apart from giving a proof for incompressible flow, the method used can be readily generalized to subsonic flow, by using a well-known theorem of analysis.

† I am indebted to the Secretary of the Aeronautical Research Committee for the opportunity of reading Whitehead's paper, entitled *Minimum Velocity Aerofoil*, 6121, Ae 2073 (dated 15 August 1942). This hitherto confidential paper will shortly appear as a Report and Memorandum of the Aeronautical Research Committee.

The conjecture made by Whitehead (loc. cit.) that Riabouchinsky's solution is very near to the true solution for compressible flow is now replaced by the proof that a certain shape defined in the hodograph plane in a very similar fashion is truly the best possible and it reduces to the solution of ref. **2** for incompressible flow. Some numerical comparisons are made for incompressible flow, but, unknown to the present author, these had already been carried out in greater detail by Whitehead. They are given for the sake of completeness (Fig. 2).

II. Two-dimensional incompressible flow

Since the aerofoils considered are symmetrical, only the upper half-plane need be considered and the flow will be taken along the x direction with stagnation points at each end on the real axis. To make the problem definite the aerofoil must be restricted to lie between two lines, say $x = \pm l$. Let an infinitesimal bulge be made at a point D on the boundary of the aerofoil. In physical terms the effect of a bulge is like a doublet with its axis tangential to the boundary at D, and with the old boundary and the real axis as given streamlines. It is evident that the flow lines due to this disturbance are such as to decrease the velocity on the original boundaries.

More precisely, mapping the upper half of the physical plane (of $z = x+iy$) on the upper half of the ζ-plane so that the aerofoil goes into the strip BC, Fig. 1, and taking the image of D in the ζ-plane as origin, let a new plane be defined by:

$$\frac{d\zeta}{dt} = (1-\epsilon^2/t^2)(1+\alpha^2\epsilon^2/t^2), \tag{1.1}$$

where ϵ is infinitesimal.

The semicircular bulge $t = \epsilon e^{i\beta}$ maps back into the ζ-plane and, in general, into the z-plane as a bulge without stagnation points, and which can be made as flat or as sharp as is desired by choosing $0 < \alpha^2 < 1$.

Let the complex potential be

$$w = U\zeta \tag{1.2}$$

for the undisturbed flow and

$$w_1 = U(t+\epsilon^2/t) \tag{1.3}$$

for the disturbed flow. Then, if Q and Q_1 denote the corresponding complex velocities

$$Q_1 = \frac{dw_1}{dz} = \frac{dw}{dz}\frac{dw_1}{dw} = Q\frac{1-\epsilon^2/t^2}{(1-\epsilon^2/t^2)(1+\alpha^2\epsilon^2/t^2)},$$

and hence

$$|Q_1| = |Q|\left|\frac{1}{1+\alpha^2\epsilon^2/t^2}\right|. \tag{1.4}$$

Except at the bulge, t is real on the boundary and therefore

$$|Q_1| < |Q|; \tag{1.5}$$

at the bulge we have

$$|Q_1| = |Q|\frac{1}{\sqrt{\{(1+\alpha^2)^2-2\alpha^2\cos 2\beta\}}} \tag{1.6}$$

where the factor varies between $1/(1+\alpha^2)$ at the ends of the bulge and $1/(1-\alpha^2)$ at the middle of the bulge. This infinitesimal variation of the boundary has the effect of

(*a*) increasing the area of the aerofoil;
(*b*) increasing, or leaving unchanged, the maximum thickness;
(*c*) decreasing the velocity except at the bulge.

If the aerofoil has either of the required properties, it follows that at D either

(i) the velocity was originally the maximum, say M, so that at the bulge $|Q_1| > M$ however small α^2, or
(ii) the chord was the maximum and D lies on $x = \pm l$.

It is concluded, therefore, that an aerofoil possessing either property (*a*) or (*b*) stated in § I must be such that at every point the velocity or chord is the maximum. This property defines an aerofoil in the hodograph plane.

III. Two-dimensional compressible flow

The same result will follow in compressible flow if it can be shown that the effect of a bulge is that of a doublet. That this is so in the subsonic case is seen as follows:

The equations of two-dimensional flow are:

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0, \tag{2.2}$$

$$\int \frac{dp}{\rho}+\tfrac{1}{2}(u^2+v^2) = \text{constant}, \tag{2.3}$$

where

$$p = p(\rho) \tag{2.4}$$

and u, v are velocity components in the x- and y-directions respectively.

Let

$$\rho u = -\psi_y \quad \text{and} \quad \rho v = \psi_x \tag{2.5}$$

where the suffixes denote differentiation with respect to x and y; (2.2) is then satisfied and (2.1) gives

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\psi_x\right)+\frac{\partial}{\partial y}\left(\frac{1}{\rho}\psi_y\right) = 0. \tag{2.6}$$

Expanding and using (2.3) and (2.4),

$$\psi_{xx}+\psi_{yy}-\frac{\dfrac{1}{q}\dfrac{d\rho}{dq}}{(\rho q)\dfrac{d}{dq}(\rho q)}[\psi_x^2\psi_{xx}+2\psi_x\psi_y\psi_{xy}+\psi_y^2\psi_{yy}] = 0, \tag{2.7}$$

where $q^2 = u^2+v^2$.

Defining the speed of sound c by $\dfrac{dp}{d\rho} = c^2$, then equation (2.7) becomes, using (2.3), (2.4), and (2.5),

$$(\psi_{xx}+\psi_{yy})[\rho^2(q^2-c^2)]-(\psi_x^2\psi_{xx}+2\psi_x\psi_y\psi_{xy}+\psi_y^2\psi_{yy}) = 0. \tag{2.8}$$

By (2.3), (2.4), (2.5), ρ and q are functions of $\psi_x^2+\psi_y^2$ only, and the equation (2.8) is of elliptic type if

$$[\rho^2(q^2-c^2)-\psi_x^2][\rho^2(q^2-c^2)-\psi_y^2]-\psi_x^2\psi_y^2 > 0, \tag{2.9}$$

or

$$\rho^2(q^2-c^2)[\rho^2(q^2-c^2)-\rho^2q^2] > 0,$$

i.e. if

$$q^2 < c^2. \tag{2.10}$$

If the bulge is now made on the boundary as in the incompressible case and α is infinitesimal the new stream function may be written

$$\psi_1 = \psi+\delta\eta, \tag{2.11}$$

where δ is a small constant.

Except for the stagnation points in the original flow the first derivatives of $\delta\eta$ are always small compared with those of ψ. Then in a region limited by the aerofoil with bulge, small circles enclosing the stagnation points, and the real axis, η satisfies an equation of the form

$$A\eta_{xx}+2B\eta_{xy}+C\eta_{yy}+E\eta_x+F\eta_y = 0. \tag{2.12}$$

Here A, B, C, E and F are known analytic functions of (x, y) because of the known analytic solution of (2.8) representing the original subsonic flow and since $AC > B^2$, by (2.8), (2.9). Also η is zero on the real axis and the boundary of the aerofoil and, since the original streamlines cross the bulge and the circles enclosing the stagnation points, is of one sign, say positive, on these parts of the boundary. Finally, on a sufficiently large semicircle η tends to zero independently of the other small quantities in the problem. By a well-known theorem (ref. **3**) the solution of (2.12) has no maximum or minimum in a region where it is regular, and it follows that η is positive in the region. Therefore on those parts of the boundary where η vanishes the normal derivative $(\partial\eta/\partial n) > 0$. By considering the behaviour near the ends of the bulge it is seen that the new velocity $(\partial\psi_1/\partial n) < (\partial\psi/\partial n)$ at all points on the boundary.

IV. Various extensions of the results

The first extension to be made is to axially symmetric subsonic flow. This may be visualized as a pseudo-two-dimensional flow in the (x,y)-plane with $y > 0$. The existence of the disturbing doublet flow has been seen to depend only on the fact that the differential equation for y is of elliptic type.

The differential equation for the stream function is

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho y}\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\rho y}\frac{\partial\psi}{\partial y}\right) = 0. \tag{3.1}$$

The second-degree terms reduce to

$$(\psi_{xx}+\psi_{yy})\rho^2q^2(1-c^2/q^2)-(1/y^2)(\psi_x^2\psi_{xx}+2\psi_x\psi_y\psi_{xy}+\psi_y^2\psi_{yy}). \tag{3.2}$$

The equation is of elliptic type if

$$[\rho^2(q^2-c^2)-\psi_x^2/y^2][\rho^2(q^2-c^2)-\psi_y^2/y^2]-\frac{\psi_x^2\psi_y^2}{y^4} > 0$$

which reduces to

$$q^2 < c^2. \tag{3.3}$$

The boundary conditions for flow past a closed body of revolution are the same as for symmetrical two-dimensional flow and so the results of §§ II and III are again true, namely that at every point the velocity is the maximum or the point lies on $x = \pm l$.

The next extension which can be made to all the preceding cases is that involving the presence of symmetrical disturbances in the stream, such as vortices or fixed bodies. For example, the relation of maximum chord or velocity applies to an aerofoil body symmetrically placed in a tunnel.

Finally, the method of proof shows that any solution obtained maximizes not only the thickness and area ratios but all functions which are additive with area, such as the moments of any order about the x-axis.

V. Solution for incompressible flow

The incompressible case is essentially the solution given by D. Riabouchinsky (ref. **2**) for a flow with free streamlines.

Let $Q = -dw/dz$; then the complex potential corresponding to equations (3) and (6) of that paper is given by

$$w = \frac{(Q+Q^{-1})/(u^{-1}-u)}{\{1+[(Q+Q^{-1})(u^{-1}-u)]^2\}^{\frac{1}{2}}}, \tag{4.1}$$

where iu is the complex velocity at infinity. The hodograph plane with streamlines is sketched in Fig. 1 and it is readily verified that equation (4.1) takes real values on the boundary.

Corresponding to equations (7) of ref. **2** the boundary is given in parametric form as

$$x(\theta) = -\frac{\sin\theta\cos\theta}{(e^2+\cos^2\theta)^{\frac{1}{2}}}+\int\limits_0^\theta \frac{\cos^2\theta\, d\theta}{(e^2+\cos^2\theta)^{\frac{1}{2}}},$$

$$y(\theta) = 4e^2\int\limits_0^1 \frac{(1-q^2)\,dq}{[(1+u^2q^2)(1+q^2/u^2)]^{\frac{3}{2}}}+e^2[-(1+e^2)^{-\frac{1}{2}}+(e^2+\cos^2\theta)^{-\frac{1}{2}}], \tag{4.2}$$

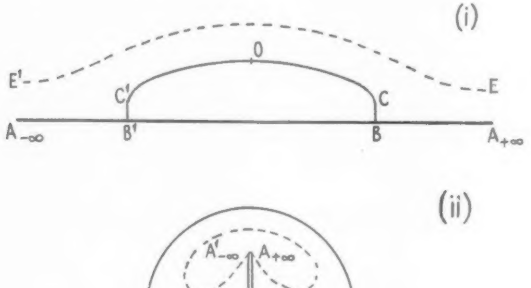

FIG. 1. (i) The physical plane, (ii) the hodograph plane.

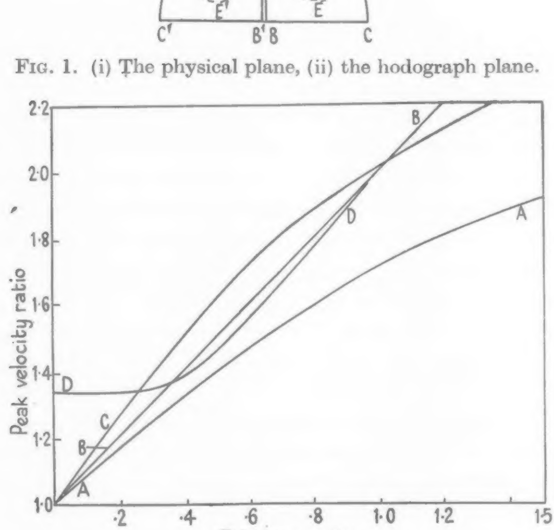

FIG. 2.

Constant velocity, AA; ellipses, BB; circular arcs, CC; Rankine ovals, DD.

where $e = \frac{1}{2}(u^{-1}-u)$. The first term of $y(\theta)$ corresponds to a straight portion at the nose, and θ is the coordinate of a point on the unit circle in the hodograph plane.

Finally, the thickness ratio, cf. equation (14) of ref. **2**, is given by Y/X, where

$$X = \{1/(1+e^2)^{\frac{1}{2}}\}\{(1+e^2)E[1/(1+e^2)^{\frac{1}{2}}]-e^2K[1/(1+e^2)^{\frac{1}{2}}]\},$$

$$Y = 4e^2\int\limits_0^1 \frac{(1-q^2)\,dq}{[(1+u^2q^2)(1+q^2/u^2)]^{\frac{1}{2}}}+e-e^2/(1+e^2)^{\frac{1}{2}} \tag{4.3}$$

and E, K are standard complete elliptic integrals.

As $e \to 0$ the peak velocity approximates to $u(1+e)$ and $X \sim 1$, $Y \sim e$. It follows that for all symmetrical aerofoils we have the following:

For a velocity peak of $u(1+e)$ where e is infinitesimal and U is the stream velocity the best thickness ratio possible is e.

VI. Calculation of the profile for subsonic flow: in particular the Kármán–Tsien case

With the notation of § II, let

$$u = \phi_x, \qquad v = \phi_y, \tag{5.1}$$

and let q and θ be the magnitude and inclination of the velocity to the y-direction. It is well known (refs. **4**, **5**) that ϕ and ψ satisfy a pair of linear equations with q, θ as independent variables.

Thus it can be easily verified from (2.1) to (2.4) that

$$\frac{1}{q}\frac{\partial\phi}{\partial q} = -\frac{d}{dq}\left(\frac{1}{\rho q}\right)\frac{\partial\psi}{\partial\theta}, \tag{5.2}$$

$$\frac{1}{\rho q}\frac{\partial\psi}{\partial q} = \frac{1}{q^2}\frac{\partial\phi}{\partial\theta}. \tag{5.3}$$

The consideration of many problems is aided by the observation that (5.2) and (5.3) can be identified with the equations of incompressible flow in a thin stratum in the plane of (q,θ). Thus, (5.2), (5.3) are identical with

$$\frac{\partial\phi}{\partial r} = -\frac{1}{hr}\frac{\partial\psi}{\partial\theta}, \tag{5.4}$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{h}\frac{\partial\psi}{\partial r}, \tag{5.5}$$

where h and r depend on q only.

Comparison of (5.2), (5.3) with (5.4), (5.5) gives

$$h = \rho(1-q^2/c^2)^{-\frac{1}{2}}, \tag{5.6}$$

$$\log r = \int \frac{dq}{\rho}(1-q^2/c^2)^{\frac{1}{2}}. \tag{5.7}$$

The usual pressure-density law can be written

$$p = k\rho^\gamma+\text{constant}, \tag{5.8}$$

from which it follows that

$$h/h_0 = \left(1-\frac{\gamma-1}{2}q^2/c_0^2\right)^{1/(\gamma-1)}\left(1-q^2/c_0^2-\frac{\gamma-1}{2}q^2\right)^{-\frac{1}{2}}, \tag{5.9}$$

$$r/r_0 = \left(\frac{t-1}{t+1}\right)^{\frac{1}{2}}\left(\frac{t+\sqrt{\beta}}{t-\sqrt{\beta}}\right)^{1/2\sqrt{\beta}}, \tag{5.10}$$

where

$$t^2 = \frac{1-\frac{1}{2}(\gamma-1)q^2/c_0^2}{1-\frac{1}{2}(\gamma+1)q^2/c_0^2}, \qquad \beta = \frac{\gamma-1}{\gamma+1}, \tag{5.11}$$

and where the suffix 0 refers to the value for $q = 0$.

For an isothermal flow ($\gamma = 1$), we have

$$r/r_0 = q/c\frac{\exp\{-\sqrt{(1-q^2/c^2)}\}}{1+\sqrt{(1-q^2/c^2)}}, \tag{5.12}$$

$$h/h_0 = (1-q^2/c^2)^{-\frac{1}{2}}e^{-\frac{1}{2}q^2/c^2}, \tag{5.13}$$

whilst the Kármán–Tsien approximation $\gamma = -1$ gives (cf. ref. **6**, **7**)

$$h/h_0 = 1, \tag{5.14}$$

$$r/r_0 = (q/c)/\{1+\sqrt{(1+q^2/c^2)}\}. \tag{5.15}$$

For a general adiabatic law the flow in the plane of (r, θ), which might be called the modified hodograph plane, is seen to be identical with the incompressible flow in a stratum of depth $h(r)$, where $h(r)$ is defined by (5.9), (5.10), (5.11).

The representation is true only for the subsonic case, which is just the case for which in § III it was shown that the required aerofoils had the simple defining property of maximum velocity or maximum chord at each point. Then,

$$\begin{aligned} d\phi &= \phi_x\,dx+\phi_y\,dy = -q\sin\theta\,dx-q\cos\theta\,dy, \\ d\psi &= \psi_x\,dx+\psi_y\,dy = \rho q\cos\theta\,dx-\rho q\sin\theta\,dy. \end{aligned} \tag{5.16}$$

Along a streamline therefore $d\psi = 0$ and

$$\begin{aligned} dx &= -d\phi\sin\theta/q, \\ dy &= -d\phi\cos\theta/q. \end{aligned} \tag{5.17}$$

If the potential satisfying equations (5.4) and (5.5) is known, these equations determine the physical boundary. Since ϕ and ψ satisfy the same boundary conditions in the subsonic case as for incompressible flow, the problem is in principle solved, e.g. by constructing the flow in an electrolyte. In the Kármán–Tsien case an explicit solution is easily derived. That this approximation is valid follows from the fact that $h(q)$ varies little over a wide range in both adiabatic and isothermal flow, though, in both cases, it eventually becomes infinite (see Table).

Table of the functions $h(q)$ and $r(q)$

Adiabatic flow				*Isothermal flow*		
q/c_0	q/q_c	h	r	q/c_0	h	r
0	0	1·00	0	0	1·00	0
·2	·22	1·00	·20	·2	1·00	·20
·4	·44	1·01	·40	·4	1·01	·38
·6	·66	1·06	·56	·6	1·03	·54
·8	·88	1·38	·69	·8	1·20	·67
·9	·99	3·50	·71	·9	1·56	—
·91	1·00	∞	·71	1·00	∞	·73

q = local fluid velocity.
c_0 = speed of sound at stagnation point.
q_c = critical speed $= c_0\sqrt{\dfrac{2}{\gamma+1}}$.
$\gamma = 1{\cdot}4$.

Consider the function

$$W = \Phi+i\Psi = \frac{(P+P^{-1})/(V^{-1}-V)}{\sqrt{\{1+[(P+P^{-1})/(V^{-1}-V)]^2\}}}, \tag{5.18}$$

where $$V = r(u)/r(M), \qquad P = re^{i\theta}/r(M) = pe^{i\theta}, \tag{5.19}$$

and M is less than $c_0\sqrt{\left(\dfrac{2}{\gamma+1}\right)}$ the critical velocity, and $r(q)$ is defined by equation (5.15). Then W satisfies the boundary conditions in the modified hodograph plane, and so the exact solution on the Kármán–Tsien approximation has been found.

Now $$\frac{dW}{dP} = \frac{\partial\Phi}{\partial P}+i\frac{\partial\Psi}{\partial P} = -\frac{i}{P}\frac{\partial\Phi}{\partial\theta}+\frac{1}{P}\frac{\partial\Psi}{\partial\theta}. \tag{5.20}$$

By integrating along the contour corresponding to the boundary of the aerofoil the following is found, corresponding to § V, equations (4.2). Write $E = \frac{1}{2}(V^{-1}-V)$; then the straight portion BC (Fig. 1) is of length

$$4E^2\int\limits_0^1 \frac{(1-p^2)p\,dp}{[(1+p^2V^2)(1+p^2/V^2)]^{\frac{3}{2}}q(p)} = Y_0 \text{ say}, \tag{5.21}$$

and putting

$$iX(\theta)-Y(\theta) = E^2\int\limits_0^{\frac{1}{2}\pi}\frac{i\sin^2\theta-\sin\theta\cos\theta}{(E^2+\cos^2\theta)^{\frac{3}{2}}}\,d\theta, \tag{5.22}$$

the maximum thickness-to-chord ratio is Y_1/X_1 where

$$Y_1 = Y(\tfrac{1}{2}\pi)+Y_0, \tag{5.23}$$

$$X_1 = X(\tfrac{1}{2}\pi). \tag{5.24}$$

For infinitesimal thickness ratio

$$X_1 \to 1 \quad\text{and}\quad Y_1 \to E.$$

Let the peak velocity $M = U(1+e)$ and therefore

$$V = \frac{r(u)}{r(M)} \sim 1 - Me\left(\frac{1}{r}\frac{dr}{dq}\right)_M,$$

and, using (5.7), $$V \sim 1 - e(1 - M^2/C_M^2)^{\frac{1}{2}} \sim 1 - E, \tag{5.25}$$
where $C_M^2 = (dp/d\rho)_M$ for the Kármán–Tsien relation between p and ρ.

For Kármán–Tsien flow the following holds: *For a velocity peak of $U(1+e)$, where U is the velocity of the stream and e is small, the best thickness ratio possible is $E = e(1-M^2/C_M^2)^{\frac{1}{2}}$.*

Conclusion

Figure 2 shows the curve of velocity peak against maximum thickness ratio for a number of well-known profiles in the case of incompressible flow. The numerical advantage over the ellipse, for example, is slight at normal thicknesses. However, since in general the effect of increasing the Mach number is to increase the 'effective' thickness ratio, it is quite likely that the advantage is really rather greater. A complete numerical investigation of the solutions for the adiabatic law with higher Mach numbers would show the highest possible Mach number attainable† with (i) given thickness/chord, (ii) given area/(chord)^2 ratios. The calculation of ψ is simpler than for the ellipse or other aerofoil shapes since the hodograph boundary is known in advance. This property in fact suggested the solution before it was realized that it had other special characteristics. As an approximation it is suggested that ψ be represented by a sum of terms, each being the imaginary part of a function of form (5.18) where $r(q)$ is any increasing function in the range $0 < q < M$; the functions being chosen to make the mean square of the error, after substitution in the differential equation for ψ, as small as possible.

Finally, I wish to acknowledge the help and encouragement of Dr. W. H. J. Fuchs in many discussions of the problem of this paper. He has pointed out that a general proof, in the hodograph plane, of the properties (*a*), (*b*) of § I is difficult because of the possibility of non-simple boundaries. A proof of (*a*) using the hodograph plane, which was found before that of §§ II, III, is therefore omitted as being less general than that given. The work itself was carried out whilst I was a member of the staff of the University College, Swansea.

† That is to say, with purely subsonic flow.

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ON THE TOTAL REFLECTION OF PLANE WAVES

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SUMMARY

The reflection and refraction of transverse plane waves at an interface parallel to the direction of polarization is considered when the incident wave is of arbitrary shape and the angle of incidence exceeds the critical angle. It is shown that the solution of this problem depends on the determination of a plane harmonic function $h(\xi, \eta)$ satisfying the condition

$$\left(\frac{\partial h}{\partial \xi}\right)_{\eta=0} - \lambda\left(\frac{\partial h}{\partial \eta}\right)_{\eta=0} = 2f'(\xi),$$

where λ is a known constant and $f'(\xi)$ a given function. By using the half-plane analogue of Poisson's formula, $h(\xi, \eta)$ can be expressed in terms of $f'(\xi)$.

The results show that the reflected and transmitted disturbances exist everywhere at all times even when the incident wave has a well-defined front, and that the transmitted disturbance due to an incident simple pulse is of the order of the reciprocal of the distance from the interface, when this distance is large.

It is pointed out that the same analysis can be applied to the treatment of the total reflection of electromagnetic waves of arbitrary shape.

Finally, the propagation of waves of arbitrary shape over the surface of a semi-infinite elastic solid is considered and shown to be possible when the velocity of propagation is that of Rayleigh waves.

THE usual treatment of the oblique reflection and refraction of plane waves which satisfy the wave equation can be worked out in terms of infinite sinusoidal wave trains or in terms of plane waves of arbitrary form. But when the angle of incidence exceeds the critical angle, the analysis, when applied to arbitrary wave forms, breaks down because the assumption that the reflected and refracted waves are proportional to the incident wave is then no longer valid. The object of this note is to set out the solution of the reflection problem in this case.

To begin with a definite example, we shall consider the refraction of transverse plane waves in an elastic solid at an interface parallel to the direction of polarization. Let the propagation vector of the incident wave lie in the xy-plane. Then the displacement, w, is at right angles to the xy-plane, and depends only on x, y, and the time t. Taking the x-axis to be the refracting interface, let a', μ' denote the velocity of propagation and the modulus of rigidity respectively for $y < 0$, and a, μ be the

corresponding quantities for $y > 0$. The displacement satisfies the wave equations

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{1}{a'^2}\frac{\partial^2 w}{\partial t^2} \quad (y<0), \tag{1 a}$$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{1}{a^2}\frac{\partial^2 w}{\partial t^2} \quad (y>0). \tag{1 b}$$

The only non-vanishing stress components are given by

$$T_{xz}=\mu\frac{\partial w}{\partial x}, \qquad T_{yz}=\mu\frac{\partial w}{\partial y} \quad (y>0);$$

$$T_{xz}=\mu'\frac{\partial w}{\partial x}, \qquad T_{yz}=\mu'\frac{\partial w}{\partial y} \quad (y<0).$$

At the interface, $y = 0$, the displacement itself and the stress component T_{yz} must be continuous, so that

$$\lim_{y\to+0}\left(\mu\frac{\partial w}{\partial y}\right)=\lim_{y\to-0}\left(\mu'\frac{\partial w}{\partial y}\right), \qquad \lim_{y\to+0} w=\lim_{y\to-0} w. \tag{2}$$

We can take the incident plane wave in the form

$$w_0=f\left(t-\frac{x\sin\gamma'+y\cos\gamma'}{a'}\right) \quad (y\leqslant 0), \tag{3}$$

where γ' is the angle of incidence. Then we can assume

$$w=w_0+w_1 \quad (y\leqslant 0); \qquad w=w_2 \quad (y\geqslant 0). \tag{4}$$

The reflected wave w_1 satisfies (1 a) and the transmitted wave w_2 satisfies (1 b). The conditions (2) become

$$w_0+w_1=w_2; \qquad \mu'\left(\frac{\partial w_0}{\partial y}+\frac{\partial w_1}{\partial y}\right)=\mu\frac{\partial w_2}{\partial y} \quad (y=0). \tag{5}$$

At $y = 0$, both w_0 and $\partial w_0/\partial y$ depend only on

$$\xi=t-\frac{x\sin\gamma'}{a'}. \tag{6}$$

This suggests that a solution can be found by assuming w_1 and w_2 to depend only on y and ξ. For the reflected wave w_1 this leads immediately to the form

$$w_1=g\left(t-\frac{x\sin\gamma'-y\cos\gamma'}{a'}\right), \tag{7}$$

where g is an as yet undetermined function. It is also a plane wave, and its propagation vector is obtained from that of the incident wave by reflection at $y = 0$. Now the velocity with which the disturbance due to the incident wave travels along the interface is

$$c=\frac{a'}{\sin\gamma'}. \tag{8}$$

If $c > a$, a solution of (1 *b*) depending on y and ξ only is simply a plane wave and the usual elementary theory of refraction results. But when the angle of incidence exceeds the critical angle, $\sin^{-1}(a'/a)$ (which can only happen when $a' < a$), the elementary treatment breaks down. This is not surprising; for in this case $c < a$, so that the transmitted movement caused by the incident wave travels ahead of the incident wave along the interface. As the incident disturbance comes on the interface from infinity, it follows that the transmitted wave must exist for all time and give rise to a reflected wave, also existing for all time, even when the incident wave has a well-defined front. The elementary theory is incapable of representing such conditions.

Considering now the case $c < a$, we can take, in accordance with our assumption,

$$w_2 = h\left\{t-\frac{x}{c},\, y\sqrt{\left(\frac{1}{c^2}-\frac{1}{a^2}\right)}\right\} = h(\xi,\eta), \text{ say.} \tag{9}$$

Substituting in (1 *b*) it follows that

$$\frac{\partial^2 h}{\partial\xi^2}+\frac{\partial^2 h}{\partial\eta^2} = 0.$$

Hence h is a plane harmonic function of ξ, η, defined for $\eta \geqslant 0$. The condition which h must satisfy at $\eta = 0$ is obtained by substituting (9) and (7) into (5), which gives

$$f(\xi)+g(\xi) = h(\xi,0) \tag{10}$$

and $$-\frac{\mu'\cos\gamma'}{a'}\{f'(\xi)-g'(\xi)\} = \mu\sqrt{\left(\frac{1}{c^2}-\frac{1}{a^2}\right)}\left(\frac{\partial h}{\partial\eta}\right)_{\eta=0}.$$

Differentiating (10) and eliminating $g'(\xi)$, it follows that

$$\left.\begin{aligned}\left(\frac{\partial h}{\partial\xi}\right)_{\eta=0}-\lambda\left(\frac{\partial h}{\partial\eta}\right)_{\eta=0} &= 2f'(\xi),\\ \text{where}\quad \lambda = \frac{a'\mu}{\mu'\cos\gamma'}\sqrt{\left(\frac{1}{c^2}-\frac{1}{a^2}\right)} &= \frac{\mu\sqrt{(a^2\sin^2\gamma'-a^2)}}{\mu' a\cos\gamma'}.\end{aligned}\right\} \tag{11}$$

As the angle of incidence, γ', increases from the critical angle to $\frac{1}{2}\pi$, the parameter λ increases from 0 to ∞.

Assume that the plane harmonic function h has the form

$$h(\xi,\eta) = \frac{1}{\pi}\operatorname{Im}\int_{-\infty}^{\infty}\frac{\phi(\alpha)+i\psi(\alpha)}{\alpha-\zeta}\,d\alpha \quad (\zeta = \xi+i\eta,\ \eta > 0). \tag{12}$$

Differentiating, it follows (using the Cauchy-Riemann equations) that

$$i\frac{\partial h}{\partial\xi}+\frac{\partial h}{\partial\eta} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi(\alpha)+i\psi(\alpha)}{(\alpha-\zeta)^2}\,d\alpha.$$

This can be transformed by partial integration. The integrated terms can be assumed to vanish, so that

$$i\frac{\partial h}{\partial \xi}+\frac{\partial h}{\partial \eta}=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{\phi'(\alpha)+i\psi'(\alpha)}{\alpha-\zeta}\,d\alpha.$$

Separating real and imaginary terms we find therefore that, for $\eta > 0$,

$$\frac{\partial h}{\partial \xi}=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{\eta\phi'(\alpha)+(\alpha-\xi)\psi'(\alpha)}{(\alpha-\xi)^2+\eta^2}\,d\alpha, \tag{12a}$$

$$\frac{\partial h}{\partial \eta}=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{(\alpha-\xi)\phi'(\alpha)-\eta\psi'(\alpha)}{(\alpha-\xi)^2+\eta^2}\,d\alpha. \tag{12b}$$

Now if $\chi(\alpha)$ is a function continuous at $\alpha=\xi$ and satisfying suitable conditions in the neighbourhood of ξ, it is known (**1**) that

$$\left.\begin{aligned}\lim_{\eta\to 0}\frac{\eta}{\pi}\int\limits_{-\infty}^{\infty}\frac{\chi(\alpha)\,d\alpha}{(\alpha-\xi)^2+\eta^2}&=\chi(\xi),\\ \lim_{\eta\to 0}\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{(\alpha-\xi)\chi(\alpha)\,d\alpha}{(\alpha-\xi)^2+\eta^2}&=\frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\frac{\chi(\alpha)}{\alpha-\xi}\,d\alpha,\end{aligned}\right\} \tag{13}$$

where the prefix P denotes that the last integral must be interpreted as a principal value, and must exist in this sense. Applying these results to $(12a,b)$ it follows that

$$\left.\begin{aligned}\left(\frac{\partial h}{\partial \xi}\right)_{\eta=0}&=\phi'(\xi)+\frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\frac{\psi'(\alpha)}{\alpha-\xi}\,d\alpha,\\ \left(\frac{\partial h}{\partial \eta}\right)_{\eta=0}&=-\psi'(\xi)+\frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\frac{\phi'(\alpha)}{\alpha-\xi}\,d\alpha.\end{aligned}\right\} \tag{14}$$

Substituting in (11),

$$\phi'(\xi)+\lambda\psi'(\xi)+\frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\frac{\psi'(\alpha)-\lambda\phi'(\alpha)}{\alpha-\xi}\,d\alpha=2f'(\xi).$$

Hence the function h given by (12) will satisfy (11) if we take

$$\phi(\xi)=\frac{2}{1+\lambda^2}f(\xi),\qquad \psi(\xi)=\frac{2\lambda}{1+\lambda^2}f(\xi),$$

whence

$$h(\xi,\eta)=\frac{2}{\pi(1+\lambda^2)}\int\limits_{-\infty}^{\infty}\frac{\eta+\lambda(\alpha-\xi)}{(\alpha-\xi)^2+\eta^2}f(\alpha)\,d\alpha. \tag{15}$$

To complete the formal solution of the reflection problem we need the value of h for $\eta = 0$. Using (13) again, it is found that

$$\lim_{\eta=0} h(\xi,\eta) = \frac{2}{1+\lambda^2} f(\xi) + \frac{2\lambda}{\pi(1+\lambda^2)} P \int_{-\infty}^{\infty} \frac{f(\alpha)}{\alpha-\xi}\, d\alpha,$$

and hence, substituting in (10), the reflected wave is obtained as

$$g(\xi) = \frac{1-\lambda^2}{1+\lambda^2} f(\xi) + \frac{2\lambda}{\pi(1+\lambda^2)} P \int_{-\infty}^{\infty} \frac{f(\alpha)}{\alpha-\xi}\, d\alpha. \tag{16}$$

The analysis by which we have obtained (15) is purely formal, but it is not difficult to state conditions on $f(\xi)$ which justify the steps by which (15) is derived. For example, we can assume $f(\xi)$ to vanish for large values of $|\xi|$ (which involves little loss of generality in a physical problem) and to possess a continuous derivative satisfying a Lipschitz condition: it then follows that h, as given by (15), is a plane harmonic function and satisfies (11).

Alternatively, we may simply substitute a given $f(\xi)$ and verify the resulting solution. The simplest example is obtained by assuming the incident wave to consist of unit displacement lasting for a finite time, say $2T$, and putting, therefore,

$$f(\xi) = 1 \quad (|\xi| < T); \qquad f(\xi) = 0 \quad (|\xi| > T). \tag{17}$$

It then follows from (15) and (16), after some simple calculations, that the reflected wave is given by

$$\left.\begin{aligned} w_1 &= g\left(t - \frac{x\sin\gamma' - y\cos\gamma'}{a'}\right), \\ g(\xi) &= \frac{1-\lambda^2}{1+\lambda^2} f(\xi) + \frac{2\lambda}{\pi(1+\lambda^2)} \log\left|\frac{\xi-T}{\xi+T}\right|, \end{aligned}\right\} \tag{18}$$

and the transmitted wave by

$$\left.\begin{aligned} w_2 &= h\left\{t - \frac{x}{c},\, y\sqrt{\left(\frac{1}{c^2}-\frac{1}{a^2}\right)}\right\}, \\ h(\xi,\eta) &= \frac{2}{\pi(1+\lambda^2)}\left\{\tan^{-1}\frac{T-\xi}{\eta} + \tan^{-1}\frac{T+\xi}{\eta} + \frac{\lambda}{2}\log\frac{(T-\xi)^2+\eta^2}{(T+\xi)^2+\eta^2}\right\}. \end{aligned}\right\} \tag{19}$$

It is easily verified that the boundary conditions hold except when

$$|\xi| = \left|t - \frac{x}{c}\right| = T,$$

and these discontinuities can be disregarded.†

† The integral theorems expressing the conservation of momentum and the continuity of deformation continue to hold near the singularities.

The logarithmic term in (18), which characterizes total reflection, gives rise to logarithmic infinities corresponding to the discontinuous beginning and end of the incident pulse. It is an odd function of ξ, and represents an additional displacement gradually increasing and becoming infinite at the instant corresponding to the arrival of the reflection of the sudden onset of the incident pulse; it then swings in the opposite sense, becoming negatively infinite at the instant corresponding to the arrival of the reflection of the discontinuous end of the incident pulse; and finally dies down, approximately proportional to the time elapsed.

The transmitted wave (19) is a disturbance propagated without change of shape along any plane $y =$ constant with velocity c. The $\tan^{-1}$ terms involving inverse tangents give an even function of ξ which can be looked upon as a distorted version of the incident pulse; the logarithmic term is an odd function of ξ which contributes an additional displacement consisting of two successive 'waves' in opposite senses. As the distance from the interface, y, increases, w_2 dies down as $1/y$. This slow rate of dying down distinguishes the total reflection of a pulse from the case of an infinite harmonic wave train, for which the disturbance falls off exponentially with penetration depth.

This solution of the problem of total reflection can also be applied to electromagnetic waves. Consider, for example, the reflection of a plane polarized wave at an interface between two non-conducting homogeneous dielectrics when the electric field is orthogonal to the plane of incidence. Taking this plane to be the xy-plane, this case arises when the only non-vanishing component of the electric field is E_z, while the magnetic field $H_z = 0$. Assuming that there is no dependence on z, Maxwell's equations in the upper half-plane are

$$\frac{\partial E_z}{\partial y}+\frac{\mu}{c}\frac{\partial H_x}{\partial t} = 0, \qquad -\frac{\partial E_z}{\partial x}+\frac{\mu}{c}\frac{\partial H_y}{\partial t} = 0,$$

$$\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}-\frac{K}{c}\frac{\partial E_z}{\partial t} = 0,$$

where K, μ are the specific inductive capacity and the magnetic permeability respectively, and c is the velocity of light in free space. We can then take

$$E_z = \frac{\partial \phi}{\partial t}, \qquad H_x = -\frac{c}{\mu}\frac{\partial \phi}{\partial y}, \qquad H_y = \frac{c}{\mu}\frac{\partial \phi}{\partial x},$$

provided that ϕ satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2} = \frac{\mu K}{c^2}\frac{\partial^2 \phi}{\partial t^2} \quad (y > 0).$$

In $y < 0$ similar equations hold, with K', μ' instead of K and μ. At $y = 0$, H_x and E_z must be continuous; hence

$$\frac{\partial\phi}{\partial t}, \qquad \frac{c}{\mu}\frac{\partial\phi}{\partial y}$$

must be continuous. The problem is therefore formally identical with that which has been solved.

The problem which has been considered is perhaps the simplest of this type. It is clear that a number of other problems involving plane waves of arbitrary shapes can be treated on the lines indicated. One example of some interest is the propagation of waves of arbitrary shape over the free surface of a semi-infinite elastic solid, analogous to Rayleigh waves.

Assuming the motion to be two-dimensional, let the solid occupy the half-plane $y \geqslant 0$. The equations of motion of an elastic solid can be satisfied by taking

$$u = \frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x}, \tag{20}$$

where u, v are the components of displacement in the x- and y-directions respectively, and ϕ, ψ satisfy the wave equations

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2}, \qquad \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = \frac{1}{b^2}\frac{\partial^2\psi}{\partial t^2},$$

a and b denoting the velocities of propagation of longitudinal and transverse waves. Now we can assume that

$$\phi = f\{x-ct, y\surd(1-c^2/a^2)\} = f(\xi,\eta), \text{ say}, \tag{21a}$$

$$\psi = g\{x-ct, y\surd(1-c^2/b^2)\} = g(\xi,\eta'), \text{ say}, \tag{21b}$$

provided that $f(\xi,\eta)$ and $g(\xi,\eta')$ are plane harmonic functions. Then the displacement at the surface, $y = 0$, is propagated without change of form with velocity c. By giving this velocity an appropriate value (which must be less than b) the boundary conditions at the surface can be satisfied.

In fact, the normal and tangential stresses which must vanish at $y = 0$ can be written in the form

$$T_{yy} = \rho\left\{(a^2-2b^2)\frac{\partial u}{\partial x}+a^2\frac{\partial v}{\partial y}\right\}, \qquad T_{yx} = \rho b^2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$

where ρ is the density. Using (20), (21 a, b) and replacing $\partial^2 f/\partial\eta^2$, $\partial^2 g/\partial\eta'^2$ by $-\partial^2 f/\partial\xi^2$, $-\partial^2 g/\partial\xi^2$ respectively, it follows that the boundary conditions are

$$T_{yy} = 2\rho b^2\left\{\left(\frac{c^2}{2b^2}-1\right)\frac{\partial^2 f}{\partial\xi^2}-\left(1-\frac{c^2}{b^2}\right)^{\frac{1}{2}}\frac{\partial^2 g}{\partial\xi\partial\eta'}\right\} = 0 \quad (\eta = \eta' = 0),$$

$$T_{xy} = 2\rho b^2\left\{\left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}\frac{\partial^2 f}{\partial\xi\partial\eta}+\left(\frac{c^2}{2b^2}-1\right)\frac{\partial^2 g}{\partial\xi^2}\right\} = 0 \quad (\eta = \eta' = 0).$$

When $\eta = \eta' = 0$ we can write η for η' and replace these equations by the equivalent conditions

$$\left(\frac{c^2}{2b^2}-1\right)f_\xi(\xi,0)-\left(1-\frac{c^2}{b^2}\right)^{\frac{1}{2}}g_\eta(\xi,0) = 0 \tag{22a}$$

$$\left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}f_\eta(\xi,0)+\left(\frac{c^2}{2b^2}-1\right)g_\xi(\xi,0) = 0, \tag{22b}$$

where partial derivatives are indicated by suffixes.

We now make use of the fact that the derivatives $h_\xi(\xi,0)$, $h_\eta(\xi,0)$ of a harmonic function are conjugate functions and constitute, if h fulfils suitable conditions, a pair of Hilbert transforms (**2**), satisfying

$$h_\xi(\xi,0) = -\frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\frac{h_\eta(\alpha,0)}{\alpha-\xi}\,d\alpha, \qquad h_\eta(\xi,0) = \frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\frac{h_\xi(\alpha,0)}{\alpha-\xi}\,d\alpha. \tag{23}$$

Formally, these relations can be derived by putting first $\phi = 0$ and then $\psi = 0$ in (14). Taking the Hilbert transform of (22 b) we obtain therefore

$$\left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}f_\xi(\xi,0)-\left(\frac{c^2}{2b^2}-1\right)g_\eta(\xi,0) = 0.$$

If this is to hold at the same time as (22 a), c must be a root of the equation

$$\left(\frac{c^2}{2b^2}-1\right)^2 = \left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}\left(1-\frac{c^2}{b^2}\right)^{\frac{1}{2}}. \tag{24}$$

This is the equation for the velocity of propagation of Rayleigh waves (**3**); it always has a positive root less than b.

Let c now be the relevant root of (24). Then, at the surface,

$$u_0(\xi) = u(\xi,0) = f_\xi(\xi,0)+\left(1-\frac{c^2}{b^2}\right)^{\frac{1}{2}}g_\eta(\xi,0),$$

$$v_0(\xi) = v(\xi,0) = \left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}f_\eta(\xi,0)-g_\xi(\xi,0).$$

By (22 a), (22 b) these equations can be replaced by

$$u_0 = \frac{c^2}{2b^2}f_\xi(\xi,0), \qquad v_0 = -\frac{c^2}{2b^2}g_\xi(\xi,0).$$

Hence, using (24), we find that the components of displacement at the free surface are linked by the equation

$$v_0(\xi) = \frac{1}{\pi}\frac{\left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}}{\frac{c^2}{2b^2}-1}P\int\limits_{-\infty}^{\infty}\frac{u_0(\alpha)}{\alpha-\xi}\,d\alpha \quad (\xi = x-ct) \tag{25}$$

and there is a corresponding equation for u_0 in terms of v_0. It is apparent

that while these surface waves are propagated with the constant velocity c and without distortion, u_0 and v_0 cannot both have a sharp wave front; there must be some displacement at all times.

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ON A SOLUTION OF THE LAMINAR BOUNDARY-LAYER EQUATION NEAR A POSITION OF SEPARATION

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SUMMARY

This paper discusses a solution of the laminar boundary-layer equations recently published by Professor S. Goldstein. Conditions near an assumed singularity at a point of separation are investigated numerically, tables are given for general use, and a comparison is made with numerical information from other sources. In addition, an improved form of the 'outer' solution of Kármán and Millikan is described.

1. Introduction

In a recent paper (**1**) Professor Goldstein has discussed an asymptotic form of solution in the neighbourhood of a separation point of a laminar boundary layer. The solution has some curious features connected with the determination of its arbitrary constants, and some doubt is, at first sight, thrown on its validity. Nevertheless, in an investigation which showed the need for a detailed examination of this problem, Hartree succeeded in obtaining very good agreement (within a range limited by the scope of his numerical results) between velocity profiles at separation given, on the one hand, by G[20]† of Professor Goldstein's paper, and on the other by his numerical solution for the whole boundary layer. In view of this agreement it seems desirable to coordinate the two investigations further, and to attempt to remove some of the uncertainty referred to above.

Interest centres first on whether the integral condition G[35] is satisfied, and secondly, on whether the disposable constant α_1 can be determined by applying the suggested condition at separation. This paper seeks to examine these questions largely by computational methods, obtaining in the process tables suitable for applications. Whilst formal proofs have seemed too difficult to attempt with existing techniques, the results of this investigation contribute strong evidence that both questions may be answered in the affirmative, and that therefore Professor Goldstein's solution is very likely to be valid.

In examining the second of the two questions, it seemed desirable to

† References in this style are to equations in Professor Goldstein's paper. Definitions and notations, which are the same as in that paper, are not repeated here.

develop an improved version of the 'outer' solution due to Kármán and Millikan, and a brief summary is given here of this extension; it is intended to discuss this more fully elsewhere.

Numerical investigation of the solution

The general plan of attack consists of computing in succession the functions f_0, f_1, f_2, f_3, f_4, and f_5—or, in the cases of f_4 and f_5, those parts of the solution which are obtainable—and then using the results, firstly, to compute values of the integrand in G[35], secondly, to calculate the velocity distribution in the neighbourhood of separation, in order to make a direct comparison with Hartree's values, and thirdly, to construct the velocity profile at separation and attempt to join it smoothly to the improved outer solution.

It is preferable not to use the explicit forms for the functions f_r deduced by Goldstein, except for the purpose of starting and checking the integrations. The method of numerical integration to be described uses the asymptotic expansions to start at some convenient large value of η, and determines f_3, f_4, f_5 by integration *towards* the origin $\eta = 0$. In this way, accumulation of error is minimized since there is no solution tending to become larger in the direction of integration, and rounding-off errors therefore die out rather than magnify themselves. Consideration of this is of some importance when each function is used in the determination of subsequent functions, and it appears in fact that, without any special precautions being taken, there is no loss in the number of significant figures as we proceed from one integration to the next.

The method, which is becoming a standard one in other connexions (especially in the making of tables to many figures, where a powerful process with automatic checking is desirable), consists essentially of a step-by-step application of a Taylor series, the coefficients of which are recalculated by a recurrence formula at each point of the integration. If h is the length of a step, and θ a variable fraction, then a function $f(\eta)$ may be expanded in the region of a point $\eta = \eta_0$ in the series

$$f(\eta_0+\theta h) = f(\eta_0)+\frac{\theta h}{1!}f'(\eta_0)+\frac{\theta^2 h^2}{2!}f''(\eta_0)+\dots.$$

For simplicity we write†

$$\frac{h^n}{n!}f^{(n)}(\eta_0) = \tau^n f(\eta_0)$$

and abbreviate the right-hand side to τ^n. Then

$$f(\eta_0+\theta h) = f(\eta_0)+\theta\tau+\theta^2\tau^2+\theta^3\tau^3+\dots.$$

† See British Association Mathematical Tables, part-volume B (*The Airy Integral*, by J. C. P. Miller), p. B7.

Also, by differentiating with respect to the independent variable θh,

$$\tau f(\eta_0+\theta h) = hf'(\eta_0+\theta h) = \tau+2\theta\tau^2+3\theta^2\tau^3+\dots,$$

$$\tau^2 f(\eta_0+\theta h) = \tfrac{1}{2}h^2 f''(\eta_0+\theta h) = \tau^2+3\theta\tau^3+6\theta^2\tau^4+\dots.$$

In applying these to the numerical integration we put $\theta = \pm 1$, obtaining

$$\left.\begin{aligned} f(\eta_0\pm h) &= f(\eta_0)\pm\tau+\tau^2\pm\tau^3+\tau^4\pm\tau^5+\dots \\ \tau f(\eta_0\pm h) &= \quad +\tau\pm 2\tau^2+3\tau^3\pm 4\tau^4+5\tau^5\pm\dots \\ \tau^2 f(\eta_0\pm h) &= \quad\quad +\tau^2\pm 3\tau^3+6\tau^4\pm 10\tau^5+\dots \end{aligned}\right\}. \tag{1.1}$$

If the function and its first two derivatives (the equations being third-order) are known at some point $\eta = \eta_0$, then values for the next step can be found by using (2.1) in conjunction with the differential equation. The formulae (2.1) with the alternative minus signs are at the same time used as a check by reproducing the values of the previous step and so testing that no significant arithmetical error has arisen in the working of that step. For ease of calculation the terms of (2.1) are grouped as follows:

$$\left.\begin{aligned} f(\eta_0\pm h) &= (\tau^0+\tau^2+\tau^4+\tau^6+\dots)\pm(\tau^1+\tau^3+\tau^5+\dots) \\ \tau f(\eta_0\pm h) &= (\tau^1+3\tau^3+5\tau^5+\dots)\pm(2\tau^2+4\tau^4+6\tau^6+\dots) \\ \tau^2 f(\eta_0\pm h) &= (\tau^2+6\tau^4+15\tau^6+\dots)\pm(3\tau^3+10\tau^5+21\tau^7+\dots) \end{aligned}\right\}. \tag{1.2}$$

The quantities in brackets are calculated separately and added or subtracted according to whether the next step or the check is required.

This method has numerous advantages in making tables, by comparison with standard finite difference methods, and even in general application in a problem such as the present one it can effect a great economy of effort. Its fundamental advantage is that it is, in principle, unlimited in accuracy when the length of step h is specified. This is not true of a difference method since high-order differences are inaccurate because of rounding-off error, and may in any case contain few figures; there need be no such loss of accuracy in the calculation of τ^n for large n, hence in general longer steps may be taken and the computation appreciably shortened. Even in cases where complexity forces the method to be curtailed to include derivatives no higher than in the differential equation—and therefore to be equivalent to a difference method—the present writer has found it more convenient on account of its very direct checking procedure.

The functions f_0, f_1, and f_2 are of course easy to tabulate directly. For convenience, in all that follows, the functions considered will be f_r/α_1^r and will still be denoted by f_r; this is effectively the same as putting $\alpha_1 = 1$, or as replacing the other variable ξ by $\alpha_1 \xi$.

2.1. The function $f_3(\eta)$

The differential equation is

$$f_3''' - \tfrac{1}{2}\eta^3 f_3'' + \tfrac{7}{2}\eta^2 f_3' - 6\eta f_3 = -10\alpha_2\,\eta^2 - \tfrac{4}{3}\eta^5 = -17{\cdot}78480\,3\eta^2 - \tfrac{4}{3}\eta^5. \tag{2.1.1}$$

Translating this into reduced derivatives we have

$$-\tau^3 = \tfrac{1}{6}h\eta^3\tau^2 - \tfrac{7}{12}h^2\eta^2\tau^1 + h^3\eta\tau^0 + h^3(2{\cdot}96413\,4\eta^2 + 0{\cdot}\dot{2}\eta^5),$$

with similar formulae, obtained by differentiation, for higher orders. On choosing a step $h = 0{\cdot}05$ and rearranging into a form more suitable for computing, the first three become

$$\begin{aligned}
-\tau^3 &= \frac{1}{960000}[20\eta\{6\tau^0 + 20\eta(-3{\cdot}5\tau^1 + 20\eta\tau^2)\}] + \\
&\qquad + 0{\cdot}00037\,05167\eta^2 + 0{\cdot}00002\,\dot{7}\eta^5, \\
-\tau^4 &= \frac{1}{3840000}[6\tau^0 + 20\eta\{-\tau^1 + 20\eta(-4\tau^2 + 60\eta\tau^3)\}] + \\
&\qquad + 0{\cdot}00000\,92629\,18\eta + 0{\cdot}00000\,1736\dot{1}\eta^4, \\
-\tau^5 &= \frac{1}{9600000}[2{\cdot}5\tau^1 + 20\eta\{-5\tau^2 + 20\eta(-1{\cdot}5\tau^3 + 120\eta\tau^4)\}] + \\
&\qquad + 0{\cdot}00000\,00926\,292 + 0{\cdot}00000\,00694\eta^3.
\end{aligned} \tag{2.1.2}$$

The asymptotic expansion of f_3 is

$$\begin{aligned}
f_3 \sim{} & -\tfrac{8}{3} + 7{\cdot}11392\,1\eta + 3{\cdot}31100\,8\eta^2 + 0{\cdot}95597\,76\eta^2(4\log\eta - 4{\cdot}60728\,6) - \\
& -\tfrac{2}{3}\eta^4 - 0{\cdot}25492\,736\left[\tfrac{1}{4}\eta^6 - \frac{15}{8}\frac{1}{\eta^2} + \frac{15}{16}\frac{1}{\eta^6} - \frac{105}{32}\frac{1}{\eta^{10}} + \dots\right].
\end{aligned} \tag{2.1.3}$$

The integration was started at $\eta = +4{\cdot}00$, using the following values computed from (2.1.3) and (2.1.2), and checked by substitution back into (2.1.1):

$$\begin{array}{lll}
f_3 = -338{\cdot}572 & \tau = -25{\cdot}3090 & \tau^2 = -0{\cdot}74695 \\
\tau^3 = -0{\cdot}01149\,0 & \tau^4 = -0{\cdot}00010\,0 & \tau^5 = -0{\cdot}00000\,02
\end{array}$$

It was taken by steps of 0·05 as far as $\eta = 0{\cdot}5$, and checks were applied at convenient points using the asymptotic expansion. Finally, a second integration, starting at $\eta = 0$ and proceeding for a short distance in the positive direction, was carried as far as $\eta = 0{\cdot}70$ with steps of 0·05, giving some overlap and check. The accumulated error after 65 steps of the main integration was found to be only 1 or 2 units in the last decimal carried for each derivative.

The functions f_0, f_1, f_2, f_3 are given, with their first derivatives, in Table 1, and it is believed that the last figure quoted is correct.

TABLE 1

η	$f_0(\eta)$	$f_0'(\eta)$	$f_1(\eta)$	$f_1'(\eta)$	$f_2(\eta)$	$f_2'(\eta)$	$f_3(\eta)$	$f_3'(\eta)$
0·0	0·000	0·000	0·00	0·0	0·000	0·000	0·00	0·00
·1	0·000	0·005	0·01	0·2	0·018	0·356	0·03	0·66
·2	0·001	0·020	0·04	0·4	0·071	0·711	0·13	1·32
·3	0·005	0·045	0·09	0·6	0·160	1·064	0·30	1·97
·4	0·011	0·080	0·16	0·8	0·284	1·414	0·53	2·61
0·5	0·021	0·125	0·25	1·0	0·442	1·758	0·82	3·22
·6	0·036	0·180	0·36	1·2	0·635	2·091	1·17	3·78
·7	0·057	0·245	0·49	1·4	0·760	2·410	1·57	4·28
·8	0·085	0·320	0·64	1·6	1·116	2·709	2·02	4·69
·9	0·122	0·405	0·81	1·8	1·401	2·983	2·50	4·97
1·0	0·167	0·500	1·00	2·0	1·712	3·224	3·01	5·12
·1	0·222	0·605	1·21	2·2	2·045	3·425	3·52	5·07
·2	0·288	0·720	1·44	2·4	2·395	3·577	4·02	4·79
·3	0·366	0·845	1·69	2·6	2·758	3·672	4·47	4·24
·4	0·457	0·980	1·96	2·8	3·127	3·699	4·85	3·35
1·5	0·562	1·125	2·25	3·0	3·495	3·648	5·13	2·08
·6	0·683	1·280	2·56	3·2	3·854	3·507	5·25	+0·35
·7	0·819	1·445	2·89	3·4	4·193	3·263	5·18	−1·90
·8	0·972	1·620	3·24	3·6	4·503	2·903	4·85	−4·77
·9	1·143	1·805	3·61	3·8	4·770	2·414	4·20	−8·33
2·0	1·333	2·000	4·00	4·0	4·981	1·781	3·16	−12·69
·1	1·544	2·205	4·41	4·2	5·120	0·987	+1·64	−17·94
·2	1·775	2·420	4·84	4·4	5·172	+0·017	−0·46	−24·21
·3	2·028	2·645	5·29	4·6	5·117	−1·147	−3·24	−31·60
·4	2·304	2·880	5·76	4·8	4·936	−2·522	−6·83	−40·27
2·5	2·604	3·125	6·25	5·0	4·605	−4·128	−11·34	−50·34
·6	2·929	3·380	6·76	5·2	4·102	−5·984	−16·95	−61·98
·7	3·281	3·645	7·29	5·4	3·399	−8·111	−23·80	−75·36
·8	3·659	3·920	7·84	5·6	2·470	−10·529	−32·08	−90·64
·9	4·065	4·205	8·41	5·8	1·283	−13·261	−42·00	−108·03
3·0	4·500	4·500	9·00	6·0	−0·194	−16·329	−53·76	−127·72
·1	4·965	4·805	9·61	6·2	−1·995	−19·757	−67·62	−149·94
·2	5·461	5·120	10·24	6·4	−4·158	−23·570	−83·84	−174·90
·3	5·990	5·445	10·89	6·6	−6·723	−27·793	−102·70	−202·86
·4	6·551	5·780	11·56	6·8	−9·731	−32·451	−124·52	−234·07
3·5	7·146	6·125	12·25	7·0	−13·228	−37·571	−149·64	−268·82
·6	7·776	6·480	12·96	7·2	−17·262	−43·182	−178·41	−307·38
·7	8·442	6·845	13·69	7·4	−21·882	−49·311	−211·25	−350·07
·8	9·145	7·220	14·44	7·6	−27·142	−55·988	−248·58	−397·20
·9	9·887	7·605	15·21	7·8	−33·099	−63·243	−290·85	−449·12
4·0	10·667	8·000	16·00	8·0	−39·811	−71·105	−338·57	−506·18

TABLE 2

η	$\bar{f}_4(\eta)$	$\bar{f}_4'(\eta)$	$\bar{f}_5(\eta)$	$\bar{f}_5'(\eta)$
0·5	−0·06	−0·31	−15·8	−35·6
·6	·11	0·63	19·5	37·9
·7	·20	1·16	23·4	40·9
·8	·35	1·99	27·7	44·8
·9	·61	3·23	32·4	49·8
1·0	−1·01	−4·97	−37·7	−56·4
·1	1·62	7·35	43·8	65·0
·2	2·51	10·51	50·8	75·9
·3	3·76	14·63	59·1	89·8
·4	5·48	19·89	68·9	107·1
1·5	−7·78	−26·50	−80·6	−128·6
·6	10·83	34·70	94·8	154·9
·7	14·78	44·77	111·8	186·8
·8	19·85	57·01	132·3	225·1
·9	26·27	71·76	157·1	270·7
2·0	−34·30	−89·40	−186·7	−324·6
·1	44·26	110·33	222·3	388·0
·2	56·49	135·01	264·7	462·0
·3	71·40	163·96	315·1	547·8
·4	89·45	197·74	374·7	646·8
2·5	−111·13	−236·96	−444·9	−760·4
·6	137·04	282·29	527·3	890·1
·7	167·82	334·45	623·6	1037·6
·8	204·18	394·25	735·5	1204·7
·9	246·95	462·56	865·2	1393·3
3·0	−297·01	−540·34	−1014·9	−1605·4
·1	355·37	628·62	1187·1	1843·0
·2	423·13	728·51	1384·5	2108·3
·3	501·50	841·23	1609·8	2403·9
·4	591·84	968·08	1866·3	2732·1
3·5	−695·64	−1110·48	−2157·4	−3095·4
·6	814·51	1269·93	2486·7	3496·8
·7	950·24	1448·07	2858·1	3939·0
·8	1104·80	1646·65	3276·0	4425·1
·9	1280·32	1867·55	3744·7	4958·2
4·0	−1479·12	−2112·78	−4269·3	−5541·7

TABLE 3

Values of $\sum (\alpha_1 \xi)^r f_r' = u_1/2\xi^2$

η \ $\alpha_1 \xi$	·04		·05		·06		·07		·08		·09	
0·0	0·000	14	0·000	16	0·000	18	0·000	21	0·000	24	0·000	26
·1	·014	23	·016	26	·018	29	·021	31	·024	33	·026	37
·2	·037	34	·042	36	·047	38	·052	41	·057	44	·063	46
·3	·071	43	·078	46	·085	49	·093	51	·101	54	·109	57
·4	·114	54	·124	56	·134	58	·144	61	·155	63	·166	66
0·5	0·168	64	0·180	66	0·192	68	0·205	71	0·218	74	0·232	76
·6	·232	73	·246	76	·260	79	·276	81	·292	83	·308	86
·7	·305	84	·322	85	·339	88	·357	90	·375	93	·394	96
·8	·389	93	·407	96	·427	98	·447	101	·468	103	·490	106
·9	·482	104	·503	106	·525	108	·548	110	·571	113	·596	115
1·0	0·586	113	0·609	115	0·633	118	0·658	120	0·684	122	0·711	124
·1	·699	123	·724	126	·751	127	·778	129	·806	132	·835	134
·2	·822	133	·850	135	·878	137	·907	140	·938	141	·969	143
·3	·955	143	·985	145	1·015	147	1·047	148	1·079	151	1·112	153
·4	1·098	153	1·130	154	1·162	157	1·195	159	1·230	159	1·265	161
1·5	1·251	163	1·284	165	1·319	166	1·354	167	1·389	169	1·426	170
·6	1·414	172	1·449	174	1·485	175	1·521	177	1·558	178	1·596	179
·7	1·586	182	1·623	183	1·660	185	1·698	186	1·736	187	1·775	187
·8	1·768	192	1·806	194	1·845	194	1·884	195	1·923	196	1·962	196
·9	1·960		2·000		2·039		2·079		2·119		2·158	
2·0	2·16	21	2·20	22	2·24	22	2·28	22	2·32	22	2·36	21
·1	2·37	22	2·42	22	2·46	22	2·50	22	2·54	22	2·57	22
·2	2·59	24	2·64	23	2·68	23	2·72	23	2·76	22	2·79	23
·3	2·83	24	2·87	24	2·91	24	2·95	24	2·98	24	3·02	23
·4	3·07	25	3·11	25	3·15	25	3·19	24	3·22	24	3·25	24
2·5	3·32	25	3·36	26	3·40	25	3·43	26	3·46	26	3·49	25
·6	3·57	27	3·62	26	3·65	27	3·69	26	3·72	26	3·74	25
·7	3·84	28	3·88	28	3·92	27	3·95	27	3·98	26	3·99	26
·8	4·12	29	4·16	29	4·19	29	4·22	28	4·24	27	4·25	26
·9	4·41	29	4·45	29	4·48	29	4·50	28	4·51	28	4·51	27
3·0	4·70	31	4·74	30	4·77	30	4·78	30	4·79	28	4·78	28
·1	5·01	32	5·04	32	5·07	30	5·08	30	5·07	29	5·06	27
·2	5·33	32	5·36	32	5·37	32	5·38	30	5·36	30	5·33	28
·3	5·65	33	5·68	32	5·69	32	5·68	32	5·66	30	5·61	29
·4	5·98	35	6·00	34	6·01	33	6·00	31	5·96	30	5·90	28
3·5	6·33	35	6·34	35	6·34	33	6·31	33	6·26	31	6·18	29
·6	6·68	36	6·69	35	6·67	35	6·64	33	6·57	31	6·47	29
·7	7·04	37	7·04	36	7·02	35	6·97	33	6·88	32	6·76	30
·8	7·41	37	7·40	37	7·37	36	7·30	34	7·20	32	7·06	29
·9	7·78		7·77		7·73		7·64		7·52		7·35	

2.2. The function $f_4(\eta)$

The differential equation is

$$f_4'''-\tfrac{1}{2}\eta^3 f_4''+4\eta^2 f_4'-7\eta f_4 = P_1-6\alpha_2^2\,\eta^2-2\alpha_2\,\eta^5+4\eta^2 f_3''-16\eta f_3'+12 f_3. \tag{2.2.1}$$

It is convenient at this point to define a new function

$$\bar{f}_4(\eta) = f_4(\eta)-\alpha_4\,\eta^2-\tfrac{1}{6}P_1(\eta^3-\tfrac{1}{105}\eta^7), \tag{2.2.2}$$

since α_4 is unknown at this stage, and the effect of P_1 may be added in later for any given problem. Then the differential equation for $\bar{f}_4$ may be written

$$\begin{aligned}\bar{f}_4'''-\tfrac{1}{2}\eta^3\bar{f}_4''+4\eta^2\bar{f}_4'-7\eta\bar{f}_4 &= -18{\cdot}97794\eta^2-3{\cdot}55696\eta^5+4\eta^2 f_3''-16\eta f_3'+12 f_3\\ &= -A_4(\eta).\end{aligned} \tag{2.2.3}$$

The successive reduced derivatives are given by

$$-\tau^3 = \frac{1}{960000}[20\eta\{7\tau^0+20\eta(-4\tau^1+20\eta\tau^2)\}]+\tfrac{1}{6}h^3 A_4(\eta),$$

$$-\tau^4 = \frac{1}{3840000}[7\tau^0+20\eta\{-\tau^1+20\eta(-5\tau^2+60\eta\tau^3)\}]+\tfrac{1}{24}h^4 A_4'(\eta),$$

where the quantities on the extreme right are connected directly with the previous integration by the relations:

$$\begin{aligned}\tfrac{1}{6}h^3 A_4(\eta) &= 0{\cdot}00039\ 53737\eta^2+0{\cdot}00007\ 41033\eta^5+\\ &\quad+(0{\cdot}0\dot{6}\eta^2\tau^2-0{\cdot}00\dot{6}\eta\tau^1+0{\cdot}00025\tau^0)(-f_3)\\ \tfrac{1}{24}h^4 A_4'(\eta) &= 0{\cdot}00000\ 98843\eta+0{\cdot}00000\ 46315\eta^4+\\ &\quad+(0{\cdot}05\eta^2\tau^3-0{\cdot}001\dot{6}\eta\tau^2-0{\cdot}00002\ 08\dot{3}\tau^1)(-f_3).\end{aligned}$$

$A_4'(\eta)$ is most easily computed from the differences of $A_4(\eta)$. The asymptotic expansion of $\bar{f}_4(\eta)$ is as follows:

$$\begin{aligned}\bar{f}_4 \sim\ & 0{\cdot}06349\ 206\eta^7-1{\cdot}18565\ 35\eta^4-5{\cdot}3\eta^3-17{\cdot}11132\ 7\eta^2+\\ &+15{\cdot}29564\ 2\eta\log\eta+9{\cdot}59998\ 56\eta-\\ &-0{\cdot}48740\ 968\left[\frac{1}{5}\eta^7-21\eta^3-\frac{21}{2}\frac{1}{\eta}+\frac{3}{2}\frac{1}{\eta^5}-\frac{315}{88}\frac{1}{\eta^9}\cdots\right]-\\ &-3{\cdot}82391\ 04\left[\frac{1}{5}\eta^5+\frac{1}{2}\frac{1}{\eta^3}-\frac{3}{4}\frac{1}{\eta^7}+\frac{35}{8}\frac{1}{\eta^{11}}\cdots\right].\end{aligned} \tag{2.2.4}$$

The integration was again started at $\eta = +4{\cdot}00$ using the following initial values:

$$\bar{f}_4 = -1479{\cdot}124,\quad \tau = -105{\cdot}6392,\quad \tau^2 = -3{\cdot}2263,\quad \tau^3 = -0{\cdot}05514,$$
$$\tau^4 = -0{\cdot}00057\ 8,\quad \tau^5 = -0{\cdot}00000\ 3,$$

and continued, as for f_3, by steps of 0·05 down to $\eta = 0{\cdot}5$. It was found that the contributions of f_3 and its derivatives to $A_4(\eta)$ were of sufficient

accuracy to give five or six significant figures in $\bar{f}_4$ where there had been five or six in f_3. This feature, which depends partly upon the correct choice of h appropriate to the accuracy required, persists in later stages, and means that the computations could, if desired, be repeated to a larger number of figures, with a smaller integration step, without any increase in complication.

2.3. The function $f_5(\eta)$

The differential equation satisfied by f_5 is

$$f_5'''-\tfrac{1}{2}\eta^3 f_5''+\tfrac{9}{2}\eta^2 f_5'-8\eta f_5 = 4f_1 f_4''-9f_1' f_4'+7f_1'' f_4+5f_2 f_3''-9f_2' f_3'+6f_2'' f_3, \tag{2.3.1}$$

and, although the right-hand side of this is known completely (apart from the term involving α_4), the complete solution for f_5 is not known. The equation (2.3.1) has three complementary functions η^2, g_5, h_5, where, by G[52] and G[53],

$$\left.\begin{aligned} g_5 &= -\sum_{m=0}^{\infty} \frac{(m-\frac{11}{4})!\,(\frac{1}{4})!}{(m+\frac{1}{4})!\,(-\frac{11}{4})!}\,\frac{1}{4m-1}\,\frac{\eta^{4m+1}}{8^m m!} \\ \text{and}\qquad h_5 &= 1+\tfrac{1}{3}\eta^4-\tfrac{1}{252}\eta^8. \end{aligned}\right\} \tag{2.3.2}$$

Thus g_5 starts with a term in η, and is the only complementary function containing exponentially large terms. h_5, on the other hand, is a terminating series, and by its use it is quite simple to reduce the order of (2.3.1) and find that the required solution with a double zero is

$$f_5(\eta) = \alpha_5\eta^2+4(\alpha_4+\alpha_2\alpha_3)(\eta-g_5)-\tfrac{1}{45}P_1\eta^6-\tfrac{1}{126}\alpha_3\eta^8+$$

$$+\eta^2\int\limits_0^\eta \left(-\frac{2}{\eta^3}+\frac{2}{3}\eta-\frac{1}{42}\eta^5\right)\int\limits_0^\eta \frac{e^{\frac{1}{8}\eta^4}\int\limits_0^\eta H_5(\eta)(-2\eta+\frac{2}{3}\eta^5-\frac{1}{42}\eta^9)e^{-\frac{1}{8}\eta^4}\,d\eta}{4-\frac{8}{3}\eta^4+\frac{34}{63}\eta^8-\frac{2}{63}\eta^{12}-\frac{1}{1764}\eta^{16}}\,d\eta d\eta, \tag{2.3.3}$$

where $H_5(\eta)$ is the part of the right-hand side of (2.3.1) which does not include the terms

$$-14(\alpha_4+\alpha_2\alpha_3)\eta^2-\tfrac{8}{3}P_1\eta^3-\tfrac{8}{3}\alpha_3\eta^5-\tfrac{4}{45}P_1\eta^7.$$

The form in which (2.3.3) is presented warrants a little discussion. There is first the complementary function term $\alpha_5\eta^2$ which in any case would have an undetermined coefficient at this stage. The second term associates the term in η of the particular integral with the non-terminating complementary function g_5: the constant α_4 is at present undetermined. The fourth term has been introduced separately since it is the only other term involving α_3 explicitly, while the remainder, represented by the repeated integral, is a function whose Taylor series for small η commences

with a term in η^4. Use of (2.3.3) to determine f_5 would be prohibitive, as a brief attempt showed. On the other hand, the method of numerical integration used so far will not, without modification, give the full solution for f_5 since no complete asymptotic expansion is known. However, the following procedure may be adopted.

The asymptotic expansion of the right-hand side of (2.3.1) is (apart from the term involving α_4) known completely. This may be substituted and the equation solved in a series of descending powers of η, starting with a term in η^6, to give an asymptotic representation of part of f_5. This will not be complete since only one of the three boundary conditions is satisfied; the asymptotic expansion so found ensures the absence of exponentially large terms, but does not necessarily correspond to a function with a double zero at the origin. It is therefore necessary to remove from it a constant and a multiple of η, and to add an arbitrary multiple of η^2 as usual.

We may conveniently define a new function

$$\bar{f}_5(\eta) = f_5(\eta) - \alpha_5\,\eta^2 - 4\alpha_4\,\eta - \lambda(1+\tfrac{1}{3}\eta^4 - \tfrac{1}{252}\eta^8) + \tfrac{1}{45}P_1\,\eta^6, \qquad (2.3.4)$$

where λ is a constant to be determined later. When this, together with the asymptotic expansions (2.1.3) and (2.2.4) is substituted in (2.3.1), we find

$$\begin{aligned} &\bar{f}_5''' - \tfrac{1}{2}\eta^3\bar{f}_5'' + \tfrac{9}{2}\eta^2\bar{f}_5' - 8\eta\bar{f}_5 \\ &\quad\sim -1{\cdot}90343\,2\eta^7 - 6{\cdot}12067\,4\eta^6 - 10{\cdot}19709\,4\eta^5\log\eta + 7{\cdot}50457\eta^5 - \\ &\qquad -23{\cdot}71307\eta^4 - 57{\cdot}10299\eta^3 - 95{\cdot}21049\eta^2\log\eta + 246{\cdot}38189\eta^2 - \\ &\qquad -61{\cdot}18257\eta\log\eta - 336{\cdot}0986\eta - 56{\cdot}91137 + 204{\cdot}7121/\eta + \\ &\qquad +66{\cdot}30731/\eta^2 - 212{\cdot}2270/\eta^3 - 163{\cdot}7696/\eta^5 - 140{\cdot}265/\eta^6 + \\ &\qquad +981{\cdot}470/\eta^7 + \dots . \end{aligned} \qquad (2.3.5)$$

The solution of this, apart from an arbitrary η^2 term (which may be taken as having been absorbed into $\alpha_5\,\eta^2$), is as follows:

$$\begin{aligned} \bar{f}_5 \sim\; &-0{\cdot}47585\,82\eta^6 - 1{\cdot}36014\,98\eta^5 - 2{\cdot}54927\,3\eta^4\log\eta + 2{\cdot}51346\eta^4 - \\ &-9{\cdot}48523\eta^3 + 27{\cdot}20300\eta\log\eta - 62{\cdot}62254\eta + 41{\cdot}26757 - \\ &-10{\cdot}23560/\eta^2 - 3{\cdot}40037\,5/\eta^3 + 5{\cdot}89519/\eta^4 - 1{\cdot}46223/\eta^6 + \\ &+5{\cdot}1006/\eta^7 - 21{\cdot}111/\eta^8 + \dots . \end{aligned} \qquad (2.3.6)$$

If an integration for $\bar{f}_5$ is started and carried back to $\eta = 0$, the resulting function will have in its Taylor series for small η a constant and a multiple of η. The complete function f_5 is constructed by adding the four terms on the right of (2.3.4), hence we see that λ is equal to the constant just mentioned, and that the coefficient of η must be equal to $4\alpha_4$. Thus f_5 could be completely determined in this way (apart from the term $\alpha_5\,\eta^2$)

were it not for the fact that the integration, by the time it reaches $\eta = 0$, has lost so many figures that the values of λ and α_4 would be practically useless. However, it will appear that the contribution to the integral G[35] of the terms involving λ and α_4 is zero, hence this difficulty is not relevant to the immediate argument. (2.3.6) was used to start an integration at $\eta = 4{\cdot}00$ with the values

$$\bar{f}_5 = -4269{\cdot}28, \quad \tau = -277{\cdot}087, \quad \tau^2 = -7{\cdot}623, \quad \tau^3 = -0{\cdot}1122,$$
$$\tau^4 = -0{\cdot}00095.$$

Only τ^3 needed calculation—τ^4 could be obtained from the differences of τ^3—and was found from the formula

$$-\tau^3 = \frac{1}{960000}[20\eta\{8\tau^0+20\eta(-4{\cdot}5\eta\tau^1+20\eta\tau^2)\}]+\tfrac{1}{6}h^3A_5(\eta),$$

where

$$\tfrac{1}{6}h^3A_5(\eta) = (\tfrac{1}{15}\eta^2\tau^2-\tfrac{3}{400}\eta\tau^1+\tfrac{7}{24000}\tau^0)(-\bar{f}_4)+$$
$$+(\tfrac{1}{12}f_2\tau^2-\tfrac{3}{800}f_2'\tau^1+\tfrac{1}{8000}f_2''\tau^0)(-f_3).$$

The integration was continued as far as $\eta = 0{\cdot}5$, and extrapolated to the origin by fitting a polynomial

$$\bar{f}_5 = a+b\eta+c\eta^4+d\eta^5+O(\eta^8)$$

which corresponds with the known form of $\bar{f}_5$ near the origin, and which gives

$$\alpha_4 = 8{\cdot}1_5, \qquad \lambda = -0{\cdot}84.$$

The functions $\bar{f}_4$ and $\bar{f}_5$ and their first derivatives are shown in Table 2.

3. The integral condition

We now come to a discussion of the condition G[108], which expresses the fact that the particular integral of G[100] which has a double zero at the origin contains no exponentially large terms. For convenience, it may be restated here as follows:

If the particular integral of

$$f_6'''-\tfrac{1}{2}\eta^3f_6''+5\eta^2f_6'-9\eta f_6 = H_6(\eta) \tag{3.1}$$

satisfies the above conditions, then

$$\int\limits_0^\infty H_6(\eta)(\eta^2-\tfrac{1}{5}\eta^6+\tfrac{1}{180}\eta^{10})e^{-\frac{1}{4}\eta^4}\,d\eta = 0. \tag{3.2}$$

Evidently, isolated terms of $H_6(\eta)$ which demonstrably do not give rise to exponentially large terms in f_6 may be dropped from the right-hand side of (3.1) and hence from the integrand of (3.2) before formulating the condition.

The form of the condition raises an immediate difficulty; to prove such an expression equal to any specified number by computational methods is, strictly speaking, impossible. At most, upper and lower bounds can be calculated, but even this has a point only when it seems likely that the expression can be proved not equal to the given number. However, we may investigate its value within the limits of accuracy of the previous computations. If we write

$$f_1 = \eta^2, \quad f_3 = \bar{f}_3 + \alpha_3\,\eta^2, \quad f_4 = \bar{f}_4 + \alpha_4\,\eta^2 + \tfrac{1}{6}P_1(\eta^3 - \tfrac{1}{105}\eta^7),$$

$$f_5 = \bar{f}_5 + \alpha_5\,\eta^2 + 4\alpha_4\,\eta + \lambda(1 + \tfrac{1}{3}\eta^4 - \tfrac{1}{252}\eta^8) - \tfrac{1}{45}P_1\,\eta^6$$

in the right-hand side of the equation for f_6, this becomes

$$-16\alpha_5\,\eta^2 - 16\alpha_4(\eta + \alpha_2\,\eta^2 + \tfrac{5}{24}\eta^5) - 8\alpha_3^2\,\eta^2 - \tfrac{26}{45}P_1\,\eta^6 - \tfrac{8}{3}\alpha_2\,P_1(\eta^3 + \tfrac{1}{20}\eta^7) +$$
$$+\lambda(16 - \tfrac{16}{3}\eta^4 - \tfrac{20}{63}\eta^8) + [4\eta^2\bar{f}_5'' - 20\eta\bar{f}_5' + 16\bar{f}_5 + 5f_2\bar{f}_4'' - 10f_2'\bar{f}_4' +$$
$$+7f_2''\bar{f}_4 + 6\bar{f}_3\bar{f}_3'' + 12\alpha_3\bar{f}_3 + 6\alpha_3\,\eta^2\bar{f}_3'' - 5\bar{f}_3'^2 - 20\alpha_3\,\eta\bar{f}_3']. \tag{3.3}$$

The particular integrals corresponding to the first few terms are:

$$4\alpha_5\,\eta + 2\alpha_3^2\,\eta + 4\alpha_4(\alpha_2\,\eta - \tfrac{1}{6}\eta^4) - \tfrac{13}{11340}P_1\,\eta^9 - \tfrac{1}{45}\alpha_2\,P_1\,\eta^6 + \tfrac{1}{6}\lambda(\eta^3 - \tfrac{1}{210}\eta^7),$$

and to give this expression a double zero, it is sufficient to add on a suitable multiple of g_6, the terminating complementary function, i.e. to add $-(4\alpha_5 + 4\alpha_2\,\alpha_4 + 2\alpha_3^2)(\eta + \tfrac{1}{15}\eta^5 - \tfrac{1}{1260}\eta^9)$. There is no constant to be removed by use of h_6 and therefore these terms induce no exponential terms in f_6. The condition (3.2) may therefore be replaced by

$$\int\limits_0^\infty J_6(\eta)(\eta^2 - \tfrac{1}{5}\eta^6 + \tfrac{1}{180}\eta^{10})e^{-\frac{1}{4}\eta^4}\,d\eta = 0, \tag{3.4}$$

where $J_6(\eta)$ is the quantity in square brackets in (3.3), which consists entirely of quantities which are known at this stage. The integrand of (3.4) may be calculated easily from the results of the numerical investigation. It is a little unsatisfactory that there is some loss of accuracy in doing this since the various groups of terms tend to cancel partially within themselves; thus, for example, at $\eta = 2$,

$$4\eta^2\bar{f}_5'' - 20\eta\bar{f}_5' + 16\bar{f}_5 = +9360 - 12990 + 2990 = -640.$$

The final product of the numerical work is thus not as accurate as might have been hoped from the integrations, which were done using six or seven figures. The effective range of integration is from about $\eta = 0{\cdot}5$ to $\eta = 3{\cdot}5$, above which the exponential term makes the integrand negligible. The form of the result is shown in Fig. 1, where the curve is

seen to have two positive and two negative regions. The areas determined by numerical integration are as follows:

$\int_{2\cdot340}^{3\cdot5} = -457\cdot4$ The fourth figure is believed to be reliable.

$\int_{1\cdot561}^{2\cdot340} = +387$ Only three figures are known here.

$\int_{1\cdot425}^{1\cdot561} = -1\cdot7$ The second figure is unreliable.

$\int_{0}^{1\cdot425} = +68$ The second figure is definitely unreliable, but should not be more than 3 or 4 units in error.

Sum $= -4$.

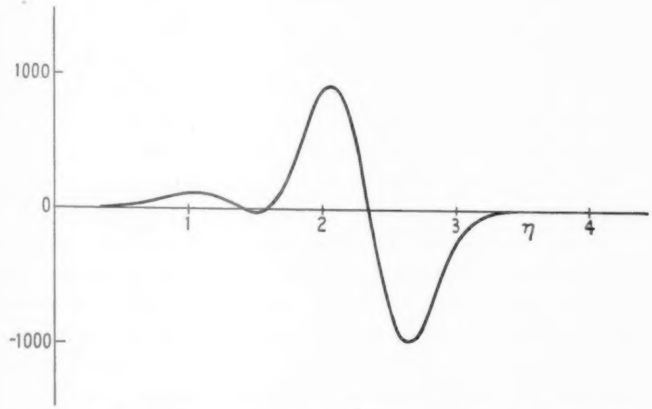

FIG. 1. Values of the integrand of $\int_0^\infty J_6(\eta)(\eta^2-\frac{1}{5}\eta^6+\frac{1}{180}\eta^{10})e^{-\frac{1}{8}\eta^4}\,d\eta$.

As regards the portion of the integral beyond 3·5, extending to infinity, the rapidly diminishing exponential makes this obviously small, but for a more formal statement we may consider the asymptotic form of the integrand. It can be shown that the part of the integrand apart from the exponential behaves for large η like η^{17}; hence if a constant A is defined so that

$$J_6(\eta)(\eta^2-\tfrac{1}{5}\eta^6+\tfrac{1}{180}\eta^{10}) < A\eta^{17} \quad (\text{all } \eta > 3\cdot5)$$

the contribution to the infinite integral is less than

$$A\int_{3\cdot5}^{\infty}\eta^{17}e^{-\frac{1}{8}\eta^4}\,d\eta = Ae^{-\frac{1}{8}(3\cdot5)^4}[2(3\cdot5)^{14}+56(3\cdot5)^{10}+1120(3\cdot5)^6+6720(3\cdot5)^2]+$$
$$+13440A\sqrt{(\tfrac{1}{2}\pi)}[1-\mathrm{erf}(3\cdot5^2/2\sqrt{2})]$$
$$= 0\cdot7A \quad (\text{approximately}).$$

A brief examination of the computed values of the integrand shows that a coefficient $A = 0{\cdot}1$ considerably over-estimates the quantity concerned, and hence the tail of the integral beyond $\eta = 3{\cdot}5$ is quite negligible to the accuracy required. Thus it seems quite likely that the condition (3.4) is satisfied, and we may proceed to a more detailed examination of the velocity field.

4. The velocity field at and near separation

The results of § 3 have contributed evidence in favour of there being a solution of the form considered. It appears likely, as Professor Goldstein has pointed out, that the solution contains just one disposable parameter —the constant α_1—and that a value for this constant may be selected by setting up a smooth transition to the main-stream conditions at the section $\xi = 0$. If this is done numerically or graphically it does not follow, of course, that the value so found is uniquely determined by the condition, but it seems most natural that it should be. Hartree, in his investigation, estimated the value of α_1 by comparing the distribution of skin friction as given by Goldstein's solution (as far as it was known at the time) with that arising from his numerical results. This indicated a value of about 0·43 for α_1. A check was applied by computing the velocity profile at separation and again comparing it with the corresponding result from the numerical integration. However, as we shall see, this result of Hartree's is in error. Also his comparison of separation profiles is of restricted value since it is confined to points quite near the wall, where the theoretical profile is remarkably insensitive to the value of α_1.

In this section we show how to derive tables for the velocity field at points near separation (the choice of α_1 being left open for the moment). A comparison is then possible between a computed velocity profile and one derived from Hartree's integration for a certain point near separation. We then point out and correct the errors in Hartree's calculations, and compare distributions of skin friction. Finally, we compare velocity profiles for the separation point derived from different sources.

The tangential velocity u_1 is given by

$$u_1 = \frac{\partial\psi_1}{\partial y_1} = 2\xi^2[f_0' + (\alpha_1\xi)f_1' + (\alpha_1\xi)^2 f_2' + \dots], \tag{4.1}$$

where, as explained previously, it is convenient to remove the constant α_1 from each function f_r and to combine it with ξ. (4.1) is in a convenient form for tabulation, and the quantity $u_1/2\xi^2$, derived from the contents of Tables 1 and 2, is shown as a function of η for several values of α_1 in Table 3. The contributions of the terms involving P_1, P_2, etc., are

negligible, and the information in Table 3 is therefore effectively independent of the pressure distribution. However, it must be pointed out that Table 3 does not include details of the velocity distribution at the separation point itself, where the effect of the first term involving P_1 is of comparable magnitude with the contribution of $\bar{f}_4$, and where the variable η becomes infinite for all y.

It is convenient here to compare the systems of coordinates used by Hartree and Goldstein. The former defines non-dimensional distances X, Y which are related to a representative velocity and length in the same way as in G[3]. The main-stream velocity at the leading edge is taken as a velocity of reference, and the representative length is chosen so that the main-stream velocity distribution is given by

$$U = 1-\tfrac{1}{8}X,$$

X being measured from the leading edge. Here, it must be noted, we have used capital letters X and Y although Hartree actually uses small letters. Then the two sets of variables (ξ, η) and (X, Y) are related in the following way:

$$\xi = \left(\frac{X_s-X}{8-X_s}\right)^{\frac{1}{4}}, \qquad \eta = \frac{1}{4}\left(\frac{8-X_s}{X_s-X}\right)^{\frac{1}{4}} Y, \tag{4.2}$$

and, from (4.1),

$$u = \tfrac{1}{4}\sqrt{\{(X_s-X)(8-X_s)\}}[f_0'+(\alpha_1\xi)f_1'+(\alpha_1\xi)^2 f_2'+...], \tag{4.3}$$

where X_s is the value of X at the separation point, determined by Hartree to be 0·9590 with an uncertainty of one or two units in the fourth decimal. The velocity u in (4.3) is Hartree's velocity and is related to Goldstein's u_1 as follows:

$$u = (1-\tfrac{1}{8}X_s)u_1 = 0{\cdot}8801u_1. \tag{4.4}$$

It is evident that if a value of X be selected, at which the velocity profile is known from Hartree's integration, then the corresponding profiles given by Goldstein's solution with various values of α_1 may be obtained quite easily from the information in Table 3. Interpolation to make the two solutions agree may give an estimate of the true value of α_1. The information given by Hartree in his report does not allow a very wide choice of points at which to try the fit. On the one hand, it is undesirable to choose a value of X very near X_s since the small uncertainty in the fourth decimal of the value 0·9590 is unnecessarily magnified, and a two-parameter fit would really be demanded. On the other hand, if X is chosen to be too small, ξ will be too large for the series in (4.3) to be adequately convergent. Considerations of this nature led to the decision that $X = 0{\cdot}956$ was the only suitable point; here $\xi = 0{\cdot}1437$, $\eta = 1{\cdot}740Y$, and $u = 0{\cdot}03633 \sum (\alpha_1\xi)^r f_r'$. In Table 4 is shown the true velocity profile

at this section together with two profiles calculated from Goldstein's solution and having $\alpha_1 = 0{\cdot}418$ and $\alpha_1 = 0{\cdot}487$ respectively. It is not obvious what is the best method of interpolating for α_1, nor is it clear how far out from the wall it is wise to proceed. That is to say, for small values of Y the solution in series must be quite accurate but the numbers involved are small and the rounding off is significant. On the other hand, for larger values of Y there are more figures but the convergence of the series solution is in doubt.

TABLE 4

Velocity profiles upstream of separation

Y	Values of u		
	True	$\alpha_1 = {\cdot}418$	$\alpha_1 = {\cdot}487$
0·0	·0000	·0000	·0000
·1	·0015	·0014	·0016
·2	·0041	·0039	·0042
·3	·0079	·0075	·0080
·4	·0127	·0122	·0129
0·5	·0186	·0180	·0188
·6	·0256	·0248	·0258
·7	·0336	·0328	·0338
·8	·0427	·0418	·0430
·9	·0528	·0519	·0532
1·0	·0639	·0630	·0644
·1	·0761	·0751	·0765
·2	·0892	·0883	·0898
·3	·1032	·1025	·1039
·4	·1182	·1176	·1189
1·5	·1340	·1337	·1350
·6	·1507	·1508	·1518
·7	·1681	·1687	·1694
·8	·1864	·1876	·1879
·9	·2053	·2073	·2071

$X = 0{\cdot}956$

$X_s = 0{\cdot}959$

It will be observed that for Y greater than about 1·5, the two profiles calculated from (4.3) do not straddle the true curve but both lie to one side of it. We may make a cautious estimate, then, that the series solution will not give four-decimal accuracy for values of Y greater than about 2.

As regards the interpolation for α_1, a little consideration suggests that the integral $\int_0^{Y_1} u\,dY$ (where Y_1 is some convenient limit) should be made equal to the value derived from Hartree's integration. If more complete data were available—i.e. for larger Y—the method would be equivalent

to making the displacement thickness equal in both cases. Taking $Y_1 = 1$ (not greater for the reasons outlined above) we find

$$\alpha_1 = 0{\cdot}47.$$

There is a small point with regard to matching velocity profiles as we have done here. The velocity as given by the series (4.3) may be in error either because α_1 is incorrect, or because the convergence is too slow for the limited number of terms available. If, as Y increases, a marked sensitivity to a change in α_1 appears before the convergence is too slow, then some conclusion may be drawn as to what is the true value of α_1. If, on the other hand, the two effects are not easily distinguishable, as in the example under consideration, any estimate of α_1 is correspondingly less reliable.

Nevertheless, the value $\alpha_1 = 0{\cdot}43$ found by Hartree seems inconsistent with the above examination. In fact, it turns out that his computations are wrong on this point, and may easily be corrected. From the formula G[22] it follows that the velocity gradient at the wall is given by

$$\left(\frac{\partial u}{\partial Y}\right)_{Y=0} = \frac{U_s}{2\sqrt{2}}\left(\frac{\partial u_1}{\partial y_1}\right)_0 = \tfrac{1}{2}U_s[2\alpha_1\xi^2+2\alpha_2\xi^3+2\alpha_3\xi^4+\ldots]$$
$$= \frac{0{\cdot}8801}{\alpha_1}[(\alpha_1\xi)^2+1{\cdot}7785(\alpha_1\xi)^3+3{\cdot}3110(\alpha_1\xi)^4+8{\cdot}15(\alpha_1\xi)^5+\ldots] \tag{4.5}$$

when numerical values are inserted. If we divide by ξ^2, and return to Hartree's variables for the right-hand side, we may write (4.5) in the form

$$100\frac{2(\partial u/\partial Y)_0}{[100(X_s-X)]^{\frac{1}{2}}} = 6{\cdot}6337[\alpha_1+0{\cdot}34526\{100(X_s-X)\}^{\frac{1}{4}}\alpha_1^2+$$
$$+0{\cdot}12478\{100(X_s-X)\}^{\frac{1}{2}}\alpha_1^3+0{\cdot}0596\{100(X_s-X)\}^{\frac{3}{4}}\alpha_1^4+\ldots], \tag{4.6}$$

where, following Hartree, certain numerical factors are introduced for convenience. Comparison of (4.6) with the corresponding equation in Hartree's report (**2**) reveals three errors; we have also an extra term which Hartree did not have. When values of $(\partial u/\partial Y)_0$ are computed from (4.5), plotted against ξ, and compared with the several isolated values found in Hartree's integration, the result is as shown in Fig. 2. It seems now that this comparison gives a value slightly greater than $0{\cdot}48$ for α_1, and the latter value is more in accord with the other estimate.

We may note that Hartree, in making his estimate, plotted not $(\partial u/\partial Y)_0$ but the quantity on the left-hand side of (4.6) which would become infinite at $X_s-X = 0$ if the wrong value of X_s were chosen. The result was that, whilst his curve was smooth for $X < 0{\cdot}950$, a peculiar irregularity was introduced by the two points for $X = 0{\cdot}956$ and $X = 0{\cdot}958$ which did

not appear to lie on a smooth continuation of the main curve. There was not therefore a very good fit between the two types of curve. The irregularity is concealed in Fig. 2 since there we are not plotting an expression which is very sensitive to the trial value of X_s. Hartree asserts that the 'bump' is real, and gives some estimates of the possible errors to support this contention, but one would feel much more confident (*a*) if there were more points between 0·950 and separation, and (*b*) if the corrections near separation in the h^2-extrapolation process were not quite so large.

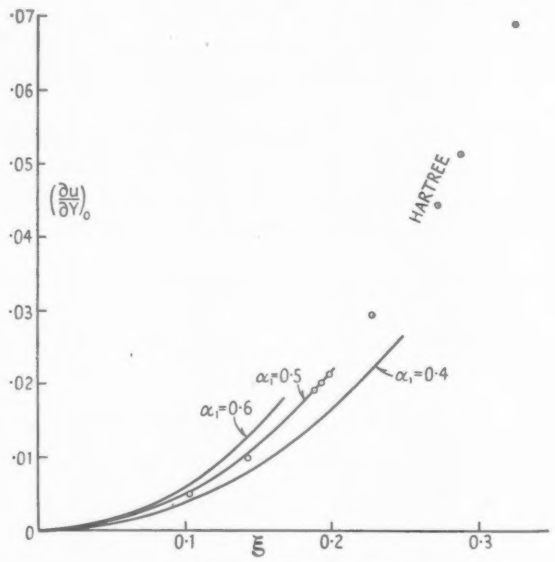

FIG. 2. Skin friction.

The shape of the velocity profile near the wall at separation is given by

$$u_1 = a_2 y_1^2 + a_3 y_1^3 + a_4 y_1^4 + \dots. \tag{4.7}$$

When we transfer to Hartree's variables and substitute the known values of the a's we find

$$u = 0{\cdot}05501Y^2 - 0{\cdot}00229\ 20\alpha_1^2 Y^4 - 0{\cdot}00065\ 73\alpha_1^3 Y^5 - \\ -(0{\cdot}00010\ 22\alpha_1^4 + 0{\cdot}00000\ 477)Y^6 + 0{\cdot}00000\ 29\alpha_1^5 Y^7 \dots, \tag{4.8}$$

or, dividing by the value of the main-stream velocity at separation,

$$\frac{u}{U} = 0{\cdot}0625Y^2 - 0{\cdot}00260\ 42\alpha_1^2 Y^4 - 0{\cdot}00074\ 69\alpha_1^3 Y^5 - \\ -(0{\cdot}00011\ 62\alpha_1^4 + 0{\cdot}00000\ 543)Y^6 + 0{\cdot}00000\ 33\alpha_1^5 Y^7 \dots. \tag{4.9}$$

The profiles given by the last formula, with various values of α_1, are shown by broken lines in Fig. 3 for the region in which $Y \geqslant 2$. They are

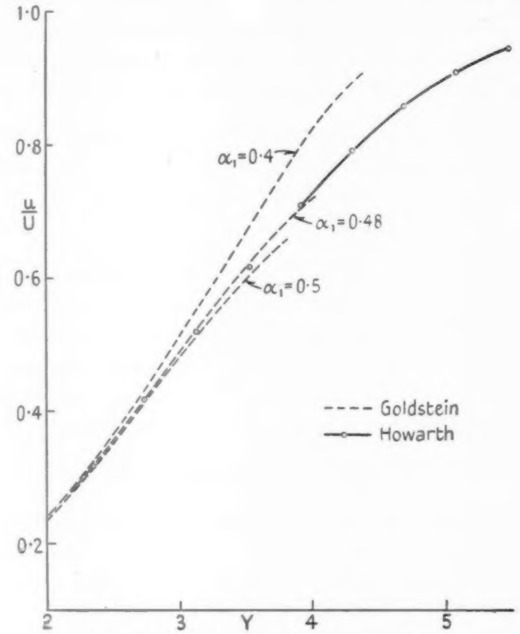

Fig. 3. Velocity profile at separation.

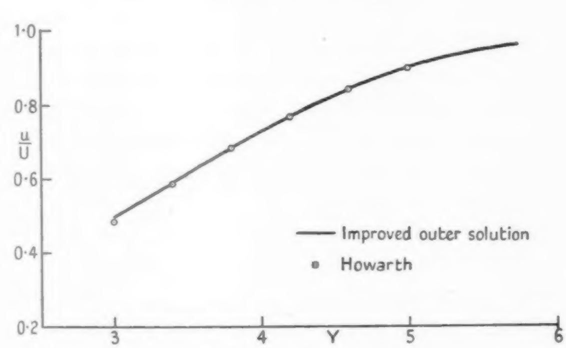

Fig. 4. Velocity profile at separation.

compared in this diagram with the separation profile given by Howarth (**3**), Table III, last column. The reliability of this column in Howarth's table was not known, but the results collected together into Figs. 4 and 5

confirm its accuracy to the number of figures involved, and make it a convenient basis of comparison. In Fig. 4 it is compared with the improved outer solution of § 5 (see later), whilst in Fig. 5 the various calculations for $Y < 1{\cdot}5$ are plotted. In this region the right-hand side of (4.9) is effectively equivalent to its first term, to three-figure accuracy, and it matters little what the precise value of α_1 is.

FIG. 5. Velocity profile at separation.

Returning now to Fig. 3, it seems that a good fit is achieved when the value of α_1 is a little greater than 0·48; this is in good agreement with the revised estimate from extrapolation of the skin friction, and in fairly good agreement with the estimate from the velocity profile at $X = 0{\cdot}956$. Professor Goldstein has suggested that the latter estimate (which gave $\alpha_1 = 0{\cdot}47$) is the most reliable of the three.

5. An improved 'outer' solution

In this final section we describe briefly an extension of the 'outer' solution of Kármán and Millikan in the particular case of a linear pressure gradient (which has been used in testing all this work). We take the boundary layer equation in the form given on p. 165 of *Modern Developments in Fluid Dynamics*, namely,

$$\frac{\partial z}{\partial \phi} = \nu\frac{u}{U}\frac{\partial^2 z}{\partial \psi^2} = \nu\left(1-\frac{z}{U^2}\right)^{\frac{1}{2}}\frac{\partial^2 z}{\partial \psi^2}, \tag{5.1}$$

where $z = U^2-u^2$, and the variable ϕ is defined to be

$$\phi = \int\limits_0^x U(x)\,dx.$$

The notation here is that of *Modern Developments*, and the variables are the natural, dimensional ones. There seems no objection to this since reference to previous literature is made easier, and the connexion with

previous sections of this paper occurs at one point only. The left-hand side of (5.1) corresponds to the inertia terms, and the right-hand side to the viscosity term, in the ordinary equation with x and y as independent variables. The first approximation, taken by Kármán and Millikan, is

$$\frac{\partial z}{\partial \phi} = \nu \frac{\partial^2 z}{\partial \psi^2} \tag{5.2}$$

and this is equivalent to an equation in which the viscosity term is multiplied by U/u. A convenient second approximation is

$$\frac{\partial z}{\partial \phi} = \nu\left(1-\frac{1}{2}\frac{z}{U^2}\right)\frac{\partial^2 z}{\partial \psi^2}, \tag{5.3}$$

and it is easily seen that this is equivalent to multiplying the viscosity term by $\frac{1}{2}(U/u+u/U)$. This quantity differs from unity by less than a given amount over a much larger range than does the ratio U/u, and the solution stands to gain appreciably in accuracy. For the case under consideration

$$U = b_0 - b_1 x, \qquad \phi = b_0 x - \tfrac{1}{2} b_1 x^2, \qquad U^2 = b_0^2 - 2b_1 \phi.$$

The solution, to Kármán and Millikan's approximation, is

$$z = b_0^2(1-\operatorname{erf}\zeta) + 2b_1\phi\left[\frac{2}{\sqrt{\pi}}\zeta e^{-\zeta^2} - (1+2\zeta^2)(1-\operatorname{erf}\zeta)\right], \tag{5.4}$$

where

$$\zeta = \frac{1}{2}\frac{\psi}{\sqrt{(\nu\phi)}}. \tag{5.5}$$

The form of (5.4) suggests solving (5.3) in the series

$$z = b_0^2[f_0(\zeta) - \lambda\phi f_1(\zeta) + \lambda^2\phi^2 f_2(\zeta) - \lambda^3\phi^3 f_3(\zeta) + - \ldots], \tag{5.6}$$

where $\lambda = 2b_1/b_0^2$. This gives the following family of differential equations:

$$(2-f_0)f_0'' + 4\zeta f_0' = 0, \tag{5.7}$$

$$(2-f_0)f_n'' + 4\zeta f_n' - 8nf_n = \sum_{s=0}^{n-1} f_s'' \sum_{r=0}^{n-s} (-1)^{n-r-s} f_r \quad (n \geqslant 1). \tag{5.8}$$

The boundary conditions are

$$f_0(0) = f_1(0) = 1; \qquad f_r(0) = 0 \quad (r \geqslant 2),$$
$$f_r(\infty) = 0 \quad (\text{all } r). \tag{5.9}$$

It is possible to simplify the numerical integrations considerably by a little analysis. Thus, if we write $2-f_0 = 2y$ in (5.7) we obtain

$$yy'' + 2\zeta y' = 0 \tag{5.10}$$

with boundary conditions $y(0) = \frac{1}{2}$, $y(\infty) = 1$. The solution of (5.10), if it exists, may be shown to be

$$y = \tfrac{1}{2}[1 + \beta\sqrt{2}\,\varphi_1(\zeta\sqrt{2}) + (\beta\sqrt{2})^2\varphi_2(\zeta\sqrt{2}) + (\beta\sqrt{2})^3\varphi_3(\zeta\sqrt{2}) + \ldots],$$

where $\beta = y'(0)$ and

$$\varphi_1(\xi) = \tfrac{1}{2}\sqrt{\pi}\,\mathrm{erf}\,\xi, \qquad \varphi_2(\xi) = \tfrac{1}{4}(1-e^{-2\xi^2}-\sqrt{\pi}\,\xi e^{-\xi^2}\,\mathrm{erf}\,\xi),$$

$$\varphi_3(\xi) = -\tfrac{1}{8}\xi e^{-\xi^2}(1+\tfrac{1}{2}e^{-2\xi^2})+\tfrac{1}{8}\sqrt{\pi}\,\mathrm{erf}\,\xi(\tfrac{1}{2}+e^{-2\xi^2}-\xi^2e^{-2\xi^2})-$$
$$-\tfrac{3}{32}\sqrt{(\tfrac{1}{3}\pi)}\mathrm{erf}(\xi\sqrt{3})+\tfrac{1}{16}\pi\xi e^{-\xi^2}(\mathrm{erf}\,\xi)^2(\tfrac{3}{2}-\xi^2).$$

Thus by taking the limits as $\xi \to \infty$,

$$\varphi_1 \to \tfrac{1}{2}\sqrt{\pi}, \qquad \varphi_2 \to \tfrac{1}{4}, \qquad \varphi_3 \to \tfrac{1}{16}\sqrt{\pi}-\tfrac{3}{32}\sqrt{(\tfrac{1}{3}\pi)},$$

we obtain an equation for β of which the first few terms are

$$1/\sqrt{2} = 0{\cdot}8862\beta+0{\cdot}3536\beta^2+0{\cdot}0297\beta^3+\dots$$

and, with an estimate of $y'(0)$ from this, the necessary trial and error consequent upon the non-linearity of the differential equation is very much reduced.

The linear equations (5.8) may be simplified in another way. For example, when $n = 1$,

$$(2-f_0)f_1''+4\zeta f_1'-(8+f_0'')f_1 = -f_0f_0'',$$

and writing $2-f_0 = 2y$ as before,

$$yf_1''+2\zeta f_1'-(4-y'')f_1 = 2y''(1-y). \tag{5.11}$$

It is found that the substitution

$$f_1 = y'g_1$$

effects a considerable simplification of (5.11), and indeed of the left-hand side of all subsequent equations in the series (5.8). We get

$$yg_1''-2\zeta g_1'-6g_1 = -4\zeta\,(1/y-1),$$

in which, not only is the left-hand side simpler, but derivatives of y are no longer involved.

In such a manner the equations may be prepared and integrated numerically. Then by substitution in (5.6) the velocity may be tabulated as a function of ψ for each value of ϕ. A simple quadrature gives y, the distance from the wall, as

$$y = \frac{1}{\sqrt{\nu}}\int\frac{d\psi}{u} = 2\sqrt{\phi}\int\frac{d\zeta}{u} \quad \text{(for each } \phi\text{)}.$$

There is, however, one arbitrary constant involved here which implies that in plotting u as a function of y an arbitrary shift in the origin of y is permissible without, however, the shape of the profile being affected.

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ON SOURCE AND VORTEX DISTRIBUTIONS IN THE LINEARIZED THEORY OF STEADY SUPERSONIC FLOW

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SUMMARY

The hyperbolic character of the differential equation satisfied by the velocity potential in linearized supersonic flow entails the presence of fractional infinities in the fundamental solutions of the equation. Difficulties arising from this fact can be overcome by the introduction of Hadamard's 'finite part of an infinite integral'. Together with the definition of certain counterparts of the familiar vector operators this leads to a natural development of the analogy between incompressible flow and linearized supersonic flow. In particular, formulae are derived for the field of flow due to an arbitrary distribution of supersonic sources and vortices.

Applications to aerofoil theory, including the calculation of the downwash in the wake of an aerofoil, are given in a separate report (ref. **9**).

1. Introduction

It is known that the elementary solution of Laplace's equation in three dimensions—i.e. the velocity potential of a source in hydrodynamics and the potential of a gravitating particle in Newtonian potential theory—has a counterpart in the linearized theory of supersonic flow, viz. the velocity potential of the so-called 'supersonic source'. However, the development of the analogy meets with obstacles which are largely due to the fact that the velocity potential of a supersonic source becomes infinite not only at the actual origin of the source but also everywhere on the rear half of the characteristic cone emanating from it. Thus, in trying to evaluate the flow across a surface surrounding a supersonic source, the resultant integral becomes infinite. This and other difficulties can be overcome by the introduction of the concept of the finite part of an infinite integral, which was first defined by Hadamard (**1**) in connexion with the solution of initial value problems for hyperbolic partial differential equations. A description of this concept is given in § 2 below, and applications to problems concerning source and doublet distributions under steady supersonic conditions will be found in §§ 3 and 4.

The concept of a vortex in the linearized theory of supersonic flow was first considered by Schlichting (**2**), who obtained the field of flow corresponding to a 'horseshoe vortex' by a synthesis of doublets. Schlichting's approach has been the subject of some criticism, as in certain respects the supersonic horseshoe vortex is different out of all recognition from its subsonic counterpart. However, it is shown in § 5 below that more generally the field of flow due to an arbitrary vorticity distribution, under steady

supersonic conditions can be calculated in strict analogy with the method due to Stokes and Helmholtz in classical hydrodynamics. The results are in agreement with Schlichting's for the particular case of a horseshoe vortex.

2. The finite part of an infinite integral

Let $D(x,y,z)$ be an algebraic function of three variables so that the equation $D(x,y,z) = 0$ determines a surface Σ in three-dimensional space. The surface Σ divides space into disconnected components Σ_n in which $D(x,y,z)$ is of constant sign; also, $D(x,y,z)$ will be supposed to change sign across any ordinary point of Σ. Further, let $f(x,y,z)$ be a real function defined in a certain region R such that

$$f(x,y,z) = g(x,y,z)+g_0(x,y,z)D^{-\frac{1}{2}n}+g_1(x,y,z)D^{-\frac{1}{2}n+1}+\dots+ \\ +g_k(x,y,z)D^{-\frac{1}{2}}+g_{k+1}(x,y,z)D^{+\frac{1}{2}}+\dots+g_{k+m}(x,y,z)D^{m-\frac{1}{2}}\dots, \tag{1}$$

where n is a positive odd integer, $n = 2k+1$, $k = 0, 1, 2,\dots$, and the functions $g(x,y,z)$, $g_0(x,y,z)$,... are either all analytic everywhere except on Σ, or at least have derivatives of a sufficiently high order, which are bounded in the neighbourhood of Σ. At any rate it is assumed that the analytic expressions for these functions may be different for the different Σ_n.

Given a small positive quantity ϵ, we denote by $N(\epsilon)$ the set of all points s which satisfy an inequality $|s-s_0| \leqslant \epsilon$ for at least one point s_0 of Σ. We denote the boundary of $N(\epsilon)$ by $B(\epsilon)$, and we denote by $R(\epsilon)$ the region obtained by excluding from R all the points of $N(\epsilon)$. Furthermore, given a curve C, a surface S, or a volume V in R, we denote by $C(\epsilon)$, $S(\epsilon)$, and $V(\epsilon)$ the subsets of C, S, and V respectively which are obtained by the exclusion of the points of $N(\epsilon)$.

The concept of the finite part $*J$ of a (finite or infinite) integral J—where J is any line, surface, or volume integral of f on C, S, or V respectively, e.g.

$$\int\limits_C f\,dx, \qquad \int\limits_S f\,dxdy, \qquad \int\limits_V f\,dxdydz,$$

where C, S, and V are supposed to be bounded—will now be defined as follows.

Given the formal expression for J, we denote by $J(\epsilon)$ the corresponding integral taken over $C(\epsilon)$, $S(\epsilon)$, or $V(\epsilon)$ only. Subject to the specified conditions of regularity, $J(\epsilon)$ will be finite and of the form

$$J(\epsilon) = a_0\,\epsilon^{-\frac{1}{2}n+1}+\dots+a_{k-1}\,\epsilon^{-\frac{1}{2}}+O(1), \tag{2}$$

where $O(1)$, as usual, denotes a function which remains finite as ϵ tends to 0. We then define $*J$ by

$$*J = \lim_{\epsilon\to 0}\,(J(\epsilon)-a_0\,\epsilon^{-\frac{1}{2}n+1}-\dots-a_{k-1}\,\epsilon^{-\frac{1}{2}}). \tag{3}$$

As stated in the introduction, the concept of the finite part of an infinite integral is due to Hadamard (**1**), whose definition, however, applies to a more restricted type of integral only. Hadamard writes $\lceil J$ instead of $*J$ which is used by Courant and Hilbert (**3**).

It will be seen that if J is finite, then $*J = J$. Also, the finite part of an (infinite) integral is invariant with respect to a transformation of coordinates, provided the Jacobian of the transformation does not vanish on Σ. In particular, if we are dealing with the finite parts of integrals involving vector quantities, the result is independent of a rotation of coordinates.

There will be no occasion for confusion if in future we refer to the finite part of a (finite or infinite) integral simply as 'a finite part'.

The finite parts of m-fold integrals in n-dimensional space, $n > 3$, $m \leqslant n$, can be defined in a strictly analogous manner. The rules valid for them are, *mutatis mutandis*, the same as for finite parts in three dimensions.

The rules of calculation with finite parts, such as the rules of addition, are the same as for ordinary integrals. Also, if f depends on a parameter λ, but D is fixed, then—provided the g functions are sufficiently regular (e.g. if they are analytic in the various Σ_n)—it is not difficult to show that we may differentiate under the sign of the integral, e.g.

$$\frac{d}{d\lambda}\left(\overset{*}{\int_C} f\,dx\right) = \overset{*}{\int_C} \frac{\partial f}{\partial \lambda}\,dx. \tag{4}$$

Under similar conditions the finite part of a multiple integral may be obtained by successive integration (including the operation of taking the finite part) with respect to the independent variables involved, taken in any arbitrary order. Thus, with the appropriate limits we have, for instance,

$$\overset{*}{\int_V} f\,dxdydz = \overset{*}{\int}\left(\overset{*}{\int}\left(\overset{*}{\int} f\,dx\right)dy\right)dz. \tag{5}$$

More generally, we shall encounter cases where D, and therefore Σ, depends algebraically on one or more parameters. We shall show (i) that even in that case we may 'differentiate under the integral sign', and (ii) that if a given integral, or finite part, involves integration with respect to such parameters, as well as with respect to one or more of the space coordinates, we may exchange the order of integration without affecting the value of the integral.

To see this we increase the ordinary three dimensions of space x, y, z by the parameter or parameters involved. Then, in the augmented space,

the surface $D = 0$ is again fixed, and in order to prove our assertions, it is sufficient to show (i)$'$ that in order to find the derivative of a finite part in n-dimensional space with respect to any variable which is not involved in the integration, we may differentiate under the sign of the integral, and (ii)$'$ that for any multiple integral in n-dimensional space, $1 < m \leqslant n$, we have

$$\overset{*}{\int}\left(\overset{*}{\int} f\,dx_1\right)dx_2...dx_m = \overset{*}{\int} f\,dx_1\,dx_2...dx_m$$

taken over the appropriate regions. It is clear that (ii)$'$ will prove (ii), by induction.

We may reduce (i)$'$ to (ii)$'$. In fact, (i)$'$ states explicitly that

$$\frac{\partial}{\partial x_m}\overset{*}{\int} f\,dx_1...dx_{m-1} = \overset{*}{\int}\frac{\partial f}{\partial x_m}dx_1...dx_{m-1},$$

and this will be proved if it can be shown that

$$\overset{*}{\int} f\,dx_1...dx_{m-1} = \overset{*}{\int}\left(\overset{*}{\int}\frac{\partial f}{\partial x_m}\,dx_1...dx_{m-1}\right)dx_m + C,$$

where the lower limit of the integral with respect to x_m is arbitrary and C is independent of x_m. Putting

$$F = \frac{\partial f}{\partial x_m},$$

we have
$$f = f_0 + \overset{*}{\int} F\,dx_m,$$

where f_0 is the value of f for an arbitrary but definite value of x_m (for given $x_1,..., x_{m-1}$), and the integral is taken with that particular value of x_m as lower limit. Now, assuming that (ii)$'$ has been proved, we have

$$\overset{*}{\int}\left(\overset{*}{\int} F\,dx_m\right)dx_1...dx_{m-1} = \overset{*}{\int}\left(\overset{*}{\int} F\,dx_1...dx_{m-1}\right)dx_m$$

and so
$$\overset{*}{\int}(f-f_0)\,dx_1...dx_m = \overset{*}{\int}\left(\overset{*}{\int}\frac{\partial f}{\partial x_m}dx_1...dx_{m-1}\right)dx_m,$$

i.e.
$$\overset{*}{\int} f\,dx_1...dx_{m-1} = \overset{*}{\int}\left(\overset{*}{\int}\frac{\partial f}{\partial x_m}dx_1...dx_{m-1}\right)dx_m + \overset{*}{\int} f_0\,dx_1...dx_m,$$

and the last term is independent of x_m, as required.

To establish (ii)′, we have to prove that

$$\overset{*}{\int}\left(\overset{*}{\int} f\,dx_m\right)dx_1...dx_{m-1} = \overset{*}{\int} f\,dx_1...dx_m. \tag{6}$$

Putting $\overset{*}{\int} f\,dx_m = F$, we see that (6) becomes

$$\overset{*}{\int\limits_S} F\,dx_1...dx_{m-1} = \overset{*}{\int\limits_R} \frac{\partial F}{\partial x_m}\,dx_1...dx_m, \tag{7}$$

taken over a certain region R on the right-hand side and over its boundary S on the left-hand side respectively. This is essentially the theorem of Gauss (or Green) for higher spaces. For $m \leqslant 3$, this theorem will be proved below for finite parts (without relying on the results of the present discussion), and the proof for greater m is quite similar.

An important example of a finite part will now be calculated. Let $D(x,y,z)$ be defined by $D \equiv x^2-\beta^2(y^2+z^2)$ and $f(x,y,z)$ by

$$\begin{cases} f(x,y,z) = \dfrac{\sigma x}{[x^2-\beta^2(y^2+z^2)]^{\frac{3}{2}}} \quad \text{for} \quad x^2 > \beta^2(y^2+z^2), \quad x > 0 \\ \text{and} \\ f(x,y,z) = 0 \text{ elsewhere,} \end{cases}$$

where σ and β are arbitrary constants. We find that all the conditions laid down at the beginning of this paragraph are satisfied in every region R not including the origin, the surface Σ being given by the cone $x^2-\beta^2(y^2+z^2) = 0$.

Further, let the open surface S be given by $x = \alpha$, $y^2+z^2 \leqslant r^2$, where $\alpha > 0$ and $r > \alpha/\beta$. S is a circular area including the circle $x = \alpha$, $y^2+z^2 = \alpha^2/\beta^2$, on which f becomes infinite of order $\frac{3}{2}$. We are going to evaluate $*J = \overset{*}{\int\limits_S} f\,dydz$.

Given $\epsilon > 0$, let $S_1(\epsilon)$ be the points of $S(\epsilon)$ for which $\alpha^2 > \beta^2(y^2+z^2)$ and $S_2(\epsilon)$ the complementary set of $S(\epsilon)$. Then f vanishes on $S_2(\epsilon)$ and so

$$J(\epsilon) = \int\limits_{S(\epsilon)} f\,dydz = \int\limits_{S_1(\epsilon)} f\,dydz + \int\limits_{S_2(\epsilon)} f\,dydz = \int\limits_{S_1(\epsilon)} f\,dydz$$

$$= \int\limits_{S_1(\epsilon)} \frac{\sigma\alpha\,dydz}{[\alpha^2-\beta^2(y^2+z^2)]^{\frac{3}{2}}} = \frac{\sigma}{\beta^2}\int\limits_0^{2\pi}\int\limits_0^{\beta\eta/\alpha} \frac{\rho\,d\rho d\theta}{(1-\rho^2)^{\frac{3}{2}}},$$

where $y = (\alpha/\beta)\rho\cos\theta$, $z = (\alpha/\beta)\rho\sin\theta$, and η is the radius of the circle bounding $S_1(\epsilon)$. It is easy to deduce from the definition of $S(\epsilon)$ that (compare Fig. 1)

$$\eta = \frac{1}{\beta}\{\alpha-\epsilon\sqrt{(1+\beta^2)}\}.$$

We then obtain

$$J(\epsilon) = \frac{2\pi\sigma}{\beta^2}\left[\frac{1}{(1-\rho^2)^{\frac{1}{2}}}\right]_0^{(\beta/\alpha)\eta} = \frac{2\pi\sigma}{\beta^2}\left\{\frac{1}{\left[\left\{2-\epsilon\frac{\sqrt{(1+\beta^2)}}{\alpha}\right\}\epsilon\frac{\sqrt{(1+\beta^2)}}{\alpha}\right]^{\frac{1}{2}}}-1\right\}$$

$$= \frac{2\pi\sigma}{\beta^2}[\alpha^{\frac{1}{2}}\{2\sqrt{(1+\beta^2)}\}^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}}-1+O(\epsilon^{\frac{1}{2}})]$$

and so

$$*J = \overset{*}{\int_S} \frac{\sigma x\, dydz}{[x^2-\beta^2(y^2+z^2)]^{\frac{3}{2}}} = -\frac{2\pi\sigma}{\beta^2}. \tag{8}$$

This result is of fundamental importance for subsequent developments.

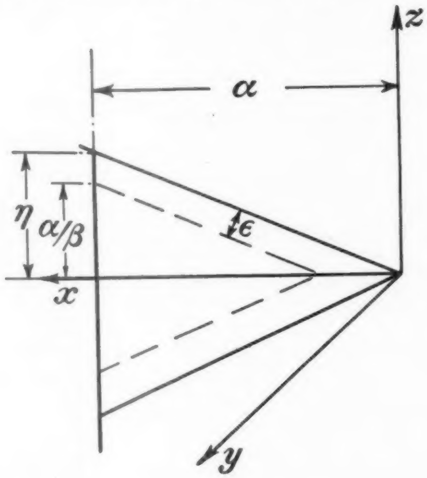

Fig. 1.

We shall also require extensions of the divergence theorem of Gauss and Green and of the curl theorem of Stokes to finite parts. Particular cases of the divergence theorem are in fact proved and applied by Hadamard in the above-mentioned book.

A function f will be called an admissible function if it satisfies the conditions laid down at the beginning of this section. A vector function $\mathbf{f}$ will be said to be admissible if its components are admissible—and this is independent of the system of coordinates. If all the first derivatives of the components of $\mathbf{f}$ are admissible, then $\operatorname{div}\mathbf{f}$ also is admissible. We are going to show that under these conditions we have for any volume V bounded by a surface S such as is considered in the ordinary divergence theorem,

$$\overset{*}{\int_S} \mathbf{f}.d\mathbf{S} = \overset{*}{\int_V} \operatorname{div}\mathbf{f}\, dV. \tag{9}$$

By equation (1), the vector $\mathbf{f} = (f_1, f_2, f_3)$ can be divided into two parts, $\mathbf{F} = (F_1, F_2, F_3)$, and $\mathbf{G} = (G_1, G_2, G_3)$, $\mathbf{f} = \mathbf{F}+\mathbf{G}$, so that the F_i are finite and continuous on Σ while the G_i become infinite there. Then $\operatorname{div}\mathbf{F}$ is either finite and continuous everywhere in its domain of definition or it becomes infinite of order $\frac{1}{2}$ on Σ. Even then $\int_V \operatorname{div}\mathbf{F}\,dV$ exists as an ordinary improper integral, and $\int_S \mathbf{F}.d\mathbf{S} = \int_V \operatorname{div}\mathbf{F}\,dV$. Since

$$\operatorname{div}\mathbf{f} = \operatorname{div}\mathbf{F}+\operatorname{div}\mathbf{G},$$

it is therefore sufficient to show that

$$\overset{*}{\int_S} \mathbf{G}.d\mathbf{S} = \overset{*}{\int_V} \operatorname{div}\mathbf{G}\,dV,$$

i.e.

$$\overset{*}{\int_S} \mathbf{G}.d\mathbf{S} - \overset{*}{\int_V} \operatorname{div}\mathbf{G}\,dV = 0. \tag{10}$$

Let $S'(\epsilon)$ be the join of (the set of points common to) V and $B(\epsilon)$. Then $V(\epsilon)$ is bounded by $S(\epsilon)+S'(\epsilon)$, the union of $S(\epsilon)$ and $S'(\epsilon)$. Hence, applying the divergence theorem to the volume $V(\epsilon)$, we obtain

$$\int_{S(\epsilon)} \mathbf{G}.d\mathbf{S} + \int_{S'(\epsilon)} \mathbf{G}.d\mathbf{S} = \int_{V(\epsilon)} \operatorname{div}\mathbf{G}\,dV$$

or

$$\int_{S(\epsilon)} \mathbf{G}.d\mathbf{S} - \int_{V(\epsilon)} \operatorname{div}\mathbf{G}\,dV = -\int_{S'(\epsilon)} \mathbf{G}.d\mathbf{S}. \tag{11}$$

Now $\int_{S(\epsilon)} \mathbf{G}.d\mathbf{S}$ is of the form

$$\int_{S(\epsilon)} \mathbf{G}.d\mathbf{S} = a_0\,\epsilon^{-(\frac{1}{2}n-1)}+\ldots+a_{k-1}\,\epsilon^{-\frac{1}{2}}+\overset{*}{\int_S} \mathbf{G}.d\mathbf{S}+H(\epsilon),$$

where $H(\epsilon)$ is a function which tends to 0 as ϵ tends to 0, and similarly $\int_{V(\epsilon)} \operatorname{div}\mathbf{G}\,dV$ will be seen to be of the form

$$\int_{V(\epsilon)} \operatorname{div}\mathbf{G}\,dV = b_0\,\epsilon^{-\frac{1}{2}n}+\ldots+b_k\,\epsilon^{-\frac{1}{2}}+\overset{*}{\int_V} \operatorname{div}\mathbf{G}\,dV+K(\epsilon),$$

where $K(\epsilon)$ is a function which tends to 0 as ϵ tends to 0.

In other words, $\int_{S(\epsilon)} \mathbf{G}.d\mathbf{S}$ and $\int_{V(\epsilon)} \operatorname{div}\mathbf{G}\,dV$ differ from $\overset{*}{\int_S} \mathbf{G}.d\mathbf{S}$ and $\overset{*}{\int_V} \operatorname{div}\mathbf{G}\,dV$ respectively only by vanishing functions of ϵ and by fractional infinities of ϵ. Hence, in order to prove (10), it is, by (11), sufficient to show that

$$\int_{S'(\epsilon)} \mathbf{G}.d\mathbf{S} = c_0\,\epsilon^{-\frac{1}{2}n}+\ldots+c_k\,\epsilon^{-\frac{1}{2}}+L(\epsilon), \tag{12}$$

where $L(\epsilon)$ is a function which tends to 0 as ϵ tends to 0. And (12) can be readily deduced from the fact that the components of $\mathbf{F}$ satisfy conditions of the type indicated by (1). In fact (1) shows that on $S'(\epsilon)$ G_1 is of the type

$$G_1 = C_0\epsilon^{-\frac{1}{2}n}+...+C_k\epsilon^{-\frac{1}{2}}, \tag{13}$$

where the C depend on the parameters of $S'(\epsilon)$, and similar expressions hold for the other components of $\mathbf{G}$.

Next, let $\mathbf{f}$ be a vector function of the same description as before, and let J be an open surface bounded by a curve C such as is considered in the ordinary curl theorem. Under these conditions we are going to show that

$$\overset{*}{\int_C} \mathbf{f}.d\mathbf{l} = \overset{*}{\int_S} \operatorname{curl}\mathbf{f}.d\mathbf{S}. \tag{14}$$

Splitting $\mathbf{f}$ into two parts $\mathbf{F}$ and $\mathbf{G}$ as before, we first show that

$$\overset{*}{\int_C} \mathbf{G}.d\mathbf{l} = \overset{*}{\int_S} \operatorname{curl}\mathbf{G}.d\mathbf{S}. \tag{15}$$

Let $C'(\epsilon)$ be the join of S and $B(\epsilon)$. Then $S(\epsilon)$ is bounded by $C(\epsilon)+C'(\epsilon)$ and so, applying Stokes's theorem to $S(\epsilon)$, we obtain

$$\int_{C(\epsilon)} \mathbf{G}.d\mathbf{l}+\int_{C'(\epsilon)} \mathbf{G}.d\mathbf{l} = \int_{S(\epsilon)} \operatorname{curl}\mathbf{G}.d\mathbf{S}. \tag{16}$$

In order to be able to deduce (15) from (16) we have to show, in the same way as in the proof of the divergence theorem, that

$$\int_{C'(\epsilon)} \mathbf{G}.d\mathbf{l} = c_0\epsilon^{-\frac{1}{2}n}+...+c_k\epsilon^{-\frac{1}{2}}+L(\epsilon), \tag{17}$$

where $\lim\limits_{\epsilon\to 0} L(\epsilon) = 0$, and this follows from (1), as before.

We still have to prove that

$$\overset{*}{\int_C} \mathbf{F}.d\mathbf{l} = \overset{*}{\int_S} \operatorname{curl}\mathbf{F}.d\mathbf{S}. \tag{18}$$

This is obvious, by Stokes's theorem, if $\operatorname{curl}\mathbf{F}$ remains finite everywhere, and if $\operatorname{curl}\mathbf{F}$ becomes infinite on Σ (in which case the right-hand side of (18) is an ordinary improper integral), provided S has not got a finite area in common with Σ. Assuming on the contrary that S has a two-dimensional subset $\bar{S}$ bounded by $\bar{C}$ in common with Σ, it is then sufficient to show that

$$\overset{*}{\int_{\bar{C}}} \mathbf{F}.d\mathbf{l} = \overset{*}{\int_{\bar{S}}} \operatorname{curl}\mathbf{F}.d\mathbf{S}. \tag{19}$$

Again, since curl $\mathbf{F}$ is an admissible vector, it follows that $\mathbf{F}$ is of the form $\mathbf{F} = \mathbf{F}^{(1)}+\mathbf{F}^{(2)}$, where the components of $\mathbf{F}^{(1)}$ are finite and continuous

on Σ and the components of $\mathbf{F}^{(2)}$ vanish on Σ (so that $|\mathbf{F}^{(2)}|\epsilon^{-\frac{1}{2}}$ remains bounded as ϵ tends to 0). Hence

$$\overset{*}{\int_{\bar{C}}} \mathbf{F}.d\mathbf{l} = \int_{\bar{C}} \mathbf{F}^{(1)}.d\mathbf{l}.$$

On the other hand, by the definition of the finite part,

$$\overset{*}{\int_{\bar{S}}} \operatorname{curl} \mathbf{F}.d\mathbf{S} = \overset{*}{\int_{\bar{S}}} \operatorname{curl} \mathbf{F}^{(1)}.d\mathbf{S},$$

and by Stokes' theorem for finite functions $\int_{\bar{C}} \mathbf{F}^{(1)}.d\mathbf{l} = \int_{\bar{S}} \operatorname{curl} \mathbf{F}^{(1)}.d\mathbf{S}$ and so $\overset{*}{\int_{\bar{C}}} \mathbf{F}.d\mathbf{l} = \overset{*}{\int_{\bar{S}}} \operatorname{curl} \mathbf{F}.d\mathbf{S}$, as required.

Equation (14) is now established completely.

3. First applications to the linearized theory of steady supersonic flow

In this and the following section we are going to discuss solutions of the equation

$$-\beta^2 \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{20}$$

in relation to the linearized theory of supersonic flow. The following details are not intended as an exhaustive introduction to that theory; their purpose merely is to establish and explain the terminology used in the sequel.

Assume that the free stream velocity of the given field of flow is parallel to the positive direction of the x-axis and is of magnitude U, where U is greater than the speed of sound

$$a = \sqrt{\frac{dp}{d\rho}}.$$

Let the total velocity components in the direction of the x-, y-, and z-axes be u, v, and w respectively; then, assuming that u is large compared with v and w, and compared with its difference from U, we obtain, for steady conditions, the linearized Eulerian equations

$$\begin{aligned} -\frac{1}{\rho}\frac{\partial p}{\partial x} &= U\frac{\partial u}{\partial x}, \\ -\frac{1}{\rho}\frac{\partial p}{\partial y} &= U\frac{\partial v}{\partial x}, \\ -\frac{1}{\rho}\frac{\partial p}{\partial z} &= U\frac{\partial w}{\partial x}, \end{aligned} \tag{21}$$

where the terms of second-order magnitude have been neglected.

Under the same assumptions, the equation of continuity which, in full, is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\frac{1}{a^2}\left(\frac{u}{\rho}\frac{\partial p}{\partial x}+\frac{v}{\rho}\frac{\partial p}{\partial y}+\frac{w}{\rho}\frac{\partial p}{\partial z}\right)=0, \tag{22}$$

becomes, taking into account (21),

$$-\beta^2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{23}$$

where $\beta^2 = M^2-1 = U^2/a^2-1$, $M = U/a$ being the Mach number. Equation (23) is the linearized equation of continuity. If, in addition, the flow is irrotational, then we have curl $\mathbf{q} = 0$, where $\mathbf{q} = (u, v, w)$, i.e.

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0, \qquad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0, \qquad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0. \tag{24}$$

In that case, there exists a velocity potential Φ so that $\mathbf{q} = -\operatorname{grad}\Phi$. Expressing u, v, w in terms of Φ in (23), we obtain (20).

Equation (22) expresses the fact that $\operatorname{div}\mathbf{q}' = 0$, where the 'current vector' $\mathbf{q}'$ is defined by $\mathbf{q}' = \rho\mathbf{q}$. Since (23), which is the linearized version of (22), indicates that $\operatorname{div}[\rho(-\beta^2 u, v, w)] = 0$, it will be seen that the corresponding 'linearized current vector' is $\mathbf{q}' = \rho(-\beta^2 u, v, w)$, where ρ is now constant. Dividing $\mathbf{q}'$ by ρ we obtain a vector $\mathbf{q}^* = (-\beta^2 u, v, w)$ which will be called the reduced current velocity, or, briefly, the c-velocity of the flow. Thus, apart from the flow of $\mathbf{q}$ across a surface S, $\int \mathbf{q}.d\mathbf{S}$, we are led to consider also the flow of $\mathbf{q}'$ across S. In order to distinguish between the two types of flow, the flow of $\mathbf{q}'$ will be called 'c-flow'. It will be seen that the linearized equation of continuity (23) is the differential expression of the fact that the total c-flow across a closed surface vanishes.

It now becomes natural to introduce alongside the conventional operators ∇, grad, div, and Δ, the operators $\nabla\mathrm{h}\,\beta$, $\operatorname{gradh}\beta$, $\operatorname{divh}\beta$, $\Delta\mathrm{h}\,\beta$. (Read 'hyperbolic nabla of index β', 'hyperbolic gradient of index β', etc. The index β will normally be fixed throughout and may therefore be omitted.) The operator $\nabla\mathrm{h}\,\beta$ will be defined by

$$\nabla\mathrm{h}\,\beta = \left(-\beta^2\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

and $\operatorname{gradh}\beta$ and $\operatorname{divh}\beta$ as the two modes of this operator which apply to scalars, and to vectors in scalar multiplication respectively. The operator $\Delta\mathrm{h}\,\beta$ is then defined by $\Delta\mathrm{h}\,\beta = \operatorname{div}\operatorname{gradh}\beta = \operatorname{divh}\beta\operatorname{grad}$. Equation (20) may now be written

$$\Delta\mathrm{h}\,\Phi = \operatorname{div}\operatorname{gradh}\Phi = \operatorname{divh}\operatorname{grad}\Phi = 0. \tag{25}$$

By the divergence theorem we have, for any function which is sufficiently regular on and inside a closed surface S bounding a volume V,

$$\int_S \operatorname{gradh}\Phi\,.\,d\mathbf{S} = \int_V \Delta\mathrm{h}\Phi\,dV. \tag{26}$$

If, in addition, Φ is a solution of (25) then

$$\int_S \operatorname{gradh}\Phi\,.\,d\mathbf{S} = 0. \tag{27}$$

However, if we replace ordinary integrals by finite parts, then equation (26) holds even when infinities are involved provided $\operatorname{gradh}\Phi$ and $\Delta\mathrm{h}\Phi$ are admissible functions with respect to a certain surface Σ. Thus, in that case,

$$\overset{*}{\int_S} \operatorname{gradh}\Phi\,.\,d\mathbf{S} = \overset{*}{\int_V} \Delta\mathrm{h}\Phi\,dV, \tag{28}$$

while (27) becomes

$$\overset{*}{\int_S} \operatorname{gradh}\Phi\,.\,d\mathbf{S} = 0. \tag{29}$$

In the sequel, such surfaces Σ will be frequently composed of cones of the form $(x-x_0)^2-\beta^2[(y-y_0)^2+(z-z_0)^2] = 0$.

We notice that if a function Ψ is admissible with respect to a surface Σ and another function Φ is regular (or has derivatives of sufficiently high order) on Σ, then the product $\Psi\Phi$ is admissible. (The product of two admissible functions is not in general admissible.) We also notice that if a finite number of functions are involved, admissible with respect to different surfaces $\Sigma^{(n)}$, then they will all be admissible with respect to one and the same surface, viz. the sum of the different $\Sigma^{(n)}$, given by the product of the functions $D^{(n)}$ defining the $\Sigma^{(n)}$.

Let Φ be a function which is regular (or has derivatives of a sufficiently high order) on and inside a volume V bounded by a surface S, and Ψ a function which, together with its first and second derivatives, is admissible in V (with respect to some algebraic surface Σ). Applying (9) to $f = \Psi\operatorname{gradh}\Phi$, we obtain

$$\overset{*}{\int_S} \Psi\operatorname{gradh}\Phi\,.\,d\mathbf{S} = \overset{*}{\int_V} \operatorname{grad}\Psi\operatorname{gradh}\Phi\,dV + \overset{*}{\int_V} \Psi\Delta\mathrm{h}\Phi\,dV, \tag{30}$$

and similarly

$$\overset{*}{\int_S} \Phi\operatorname{gradh}\Psi\,.\,d\mathbf{S} = \overset{*}{\int_V} \operatorname{grad}\Phi\operatorname{gradh}\Psi\,dV + \overset{*}{\int_V} \Phi\Delta\mathrm{h}\Psi\,dV. \tag{31}$$

Now

$$\operatorname{grad}\Psi\operatorname{gradh}\Phi = -\beta^2\frac{\partial\Psi}{\partial x}\frac{\partial\Phi}{\partial x}+\frac{\partial\Psi}{\partial y}\frac{\partial\Phi}{\partial y}+\frac{\partial\Psi}{\partial z}\frac{\partial\Phi}{\partial z} = \operatorname{gradh}\Psi\operatorname{grad}\Phi.$$

Hence, subtracting (31) from (30), we obtain

$$\int\limits_S^{*} (\Psi \operatorname{gradh}\Phi - \Phi \operatorname{gradh}\Psi).d\mathbf{S} = \int\limits_V^{*} (\Psi \Delta\mathrm{h}\,\Phi - \Phi\Delta\mathrm{h}\,\Psi)\,dV. \qquad (32)$$

This is the counterpart of Green's formula, extended, however, to include finite parts (compare refs. **1** and **3**).

4. Source and doublet distributions in steady supersonic flow

Elementary solutions of equations (20) or (25) are the functions $\Phi_P(x,y,z)$ defined by

$$\left.\begin{aligned} \Phi_P(x,y,z) &= \frac{\sigma}{\sqrt{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]}} \\ &\quad \text{for} \quad (x-x_0)^2 > \beta^2[(y-y_0)^2+(z-z_0)^2], \quad x > x_0 \\ \text{and} \quad \Phi_P(x,y,z) &= 0 \quad \text{elsewhere,} \end{aligned}\right\} \qquad (33)$$

where $P = (x_0, y_0, z_0)$ and σ are arbitrary. Φ_P will be said to be the velocity potential of a source of strength σ located (or, 'with origin') at P. The actual velocity potential of a source travelling with a velocity $-U$, in a frame of reference travelling with the source, is obtained by adding $-Ux$ to Φ_P as given by (33). For reasons of simplicity, Φ_P as in (33) will be called a source for positive as well as for negative σ.

Similarly, a function Ψ_P will be said to be the velocity potential of a counter-source of strength σ located at P if it is given by

$$\left.\begin{aligned} \Psi_P(x,y,z) &= \frac{\sigma}{\sqrt{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]}} \\ &\quad \text{for} \quad (x-x_0)^2 > \beta^2[(y-y_0)^2+(z-z_0)^2], \quad x < x_0 \\ \text{and} \quad \Psi_P(x,y,z) &= 0 \quad \text{elsewhere.} \end{aligned}\right\} \qquad (34)$$

A 'doublet' is obtained, by definition, by differentiating Φ_P with respect to length in any given direction, the differentiation being performed relative to the coordinates of P. Thus the velocity potential $\overline{\Phi}_P$ of a doublet whose 'axis' is parallel to the z-axis is given by

$$\left.\begin{aligned} \overline{\Phi}_P(x,y,z) &= \frac{-\sigma\beta^2(z-z_0)}{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]^{\frac{3}{2}}} \\ &\quad \text{for} \quad (x-x_0)^2 > \beta^2[(y-y_0)^2+(z-z_0)^2], \quad x > x_0 \\ \text{and} \quad \overline{\Phi}_P(x,y,z) &= 0 \quad \text{elsewhere.} \end{aligned}\right\} \qquad (35)$$

A 'counter-doublet' is obtained, by definition, by applying a similar operation to Ψ_P. An asterisk will be employed to indicate fundamental solutions of unit strength (e.g. Φ_P^*).

It will be seen that the velocity potentials of sources, counter-sources, doublets, etc., and all their derivatives are admissible functions in all regions excluding their respective origins, the surface Σ being given by

$$(x-x_0)^2-\beta^2[(y-y_0)^2+(z-z_0)^2] = 0.$$

Also, the potentials of sources and counter-sources tend to zero of order $\frac{1}{2}$ as the affix tends to infinity in any direction not asymptotic to Σ, and similarly doublets and counter-doublets tend to zero of order $\frac{3}{2}$ under the same conditions.

The velocity potentials due to line, surface, and volume distributions in points outside the distributions are obtained by evaluating the integrals $\int \sigma\Phi_P^*\,dl$, $\int \sigma\Phi_P^*\,dS$, $\int \sigma\Phi_P^*\,dV$, where σ denotes the (variable) line, surface, or volume density, and Φ_P^* denotes the velocity potential of a source of unit strength the coordinates of whose origin coincide with the variable(s) of integration. For sufficiently regular distribution functions σ (e.g. if σ has continuous bounded first derivatives) these integrals exist as ordinary improper integrals. For instance, for a surface distribution we obtain

$$\Phi(x,y,z) = \int \frac{\sigma\,dS}{\sqrt{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]}}, \tag{36}$$

where σ is defined as a function of the parameters u, v, of the surface S, given by $x_0 = x_0(u,v)$, $y_0 = y_0(u,v)$, $z_0 = z_0(u,v)$ and the integral is taken over those parts of the surface for which

$$(x-x_0)^2 > \beta^2[(y-y_0)^2+(z-z_0)^2] \quad \text{and} \quad x > x_0.$$

The position is different in the case of line, surface, or volume distributions of doublets, since the integrals corresponding to such distributions, viz. $\int \sigma\Phi_P^*\,dl$, $\int \sigma\Phi_P^*\,dS$, and $\int \sigma\Phi_P^*\,dV$, are in general infinite. Thus, for a surface distribution of doublets whose axes are all parallel to the z-axis we obtain

$$\int \frac{-\sigma(z-z_0)\beta^2\,dS}{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]^{\frac{3}{2}}} \tag{37}$$

which is, in general, infinite. However, provided σ is sufficiently regular, the finite part of the integral still exists, and we may say that

$$\Phi = \overset{*}{\int} \sigma\Phi_P^*\,dS$$

is the potential due to a doublet distribution over S (with similar definitions for potentials due to line or volume distributions). An alternative method which avoids the use of the finite part and which has been used by Schlichting (**2**) and others, is to consider first the corresponding integral for sources (eqn. (36)) and then to differentiate with respect to $(-z)$.

From a physical point of view this means that we calculate the potential due to two infinitely near source distributions of opposite strength. The final result is the same since, according to the rules given in the preceding section, finite parts can always be differentiated under the sign of the integral. It is precisely the possibility of carrying out all the necessary operations directly, without fear of obtaining meaningless symbols, which makes the finite part such a convenient concept. It will be seen that the alternative method is applicable only when all the doublets have parallel axes.

In order to define the potential due to a volume distribution of sources (or of counter-sources) we have, as in classical theory, to take recourse to a limit process, as the integrand $\sigma\Phi_P$ tends to ∞ of the order 1 on approaching the point for which the potential is calculated. We therefore surround the point by a small sphere of radius ϵ, evaluate the integral excluding the interior of the sphere, and then let ϵ tend to 0. For finite ϵ, the integrals in question exist as ordinary improper integrals, and the limit exists, as ϵ tends to 0, since the volume of the sphere tends to 0 as ϵ^3.

We are now going to show that the c-flow—defined as a finite part—across a closed surface surrounding a source of strength σ is equal to $2\pi\sigma$.

We have to prove

$$*J = -\overset{*}{\int\limits_S} \operatorname{gradh}\Phi_P . d\mathbf{S} = 2\pi\sigma, \tag{38}$$

where Φ_P is defined by (33).

We may simplify the problem without loss of generality by assuming $x_0 = y_0 = z_0 = 0$. Now let S be a small cylindrical surface bounded by two planes $x = \pm\alpha$ and by the cylinder $y^2+z^2 = r^2$, where $r > \alpha/\beta$. Then the integrand of (35) vanishes everywhere on S except in the circular area belonging to the plane $x = \alpha$. Hence $*J$ reduces to

$$*J = \overset{*}{\int\limits_{y^2+z^2<r^2}} \frac{\sigma\alpha\beta^2\, dydz}{[\alpha^2-\beta^2(y^2+z^2)]^{\frac{3}{2}}},$$

and therefore $*J = 2\pi\sigma$, by (8). This confirms the theorem for the particular case of a circular cylinder.

Next, let S be an arbitrary surface surrounding the source, then we may find a small cylindrical surface S' of the above description inside S and we only have to show that

$$\overset{*}{\int\limits_S} \operatorname{gradh}\Phi_P . d\mathbf{S} = \overset{*}{\int\limits_{S'}} \operatorname{gradh}\Phi_P . d\mathbf{S}.$$

Let V be the volume bounded by S and S'. Then by the divergence theorem for finite parts, (9),

$$\overset{*}{\int\limits_S} \operatorname{gradh}\Phi_P.d\mathbf{S} - \overset{*}{\int\limits_S} \operatorname{gradh}\Phi.d\mathbf{S} = \overset{*}{\int\limits_V} \operatorname{div}\operatorname{gradh}\Phi_P\,dV.$$

But
$$\overset{*}{\int\limits_V} \operatorname{div}\operatorname{gradh}\Phi_P\,dV = \overset{*}{\int\limits_V} \Delta\mathrm{h}\Phi_P\,dV = 0,$$

since Φ satisfies (25) and V does not include the origin. This proves that (38) is true generally.

Similarly, we obtain for counter-sources, whose potential Ψ_P is given by (34),

$$-\overset{*}{\int\limits_S} \operatorname{gradh}\Psi_P.d\mathbf{S} = 2\pi\sigma. \tag{39}$$

More generally, if a surface S surrounds a finite number of sources of strengths σ_n superimposed on an arbitrary field of flow which is regular inside S, then

$$-\overset{*}{\int\limits_S} \operatorname{gradh}\Phi.d\mathbf{S} = 2\pi\sum\sigma_n. \tag{40}$$

There is a similar theorem for counter-sources.

In fact, (40) follows immediately from (29) and (38).

We may also deduce, taking into account (i) early in § 2, that the c-flow across a surface surrounding a doublet vanishes, and more generally that the flow across a closed surface is not affected by the superposition of doublets either inside it or outside.

Finally, it follows from (ii) early in § 2, that (38) can also be applied to a continuous distribution of sources inside a surface S, so that

$$-\overset{*}{\int\limits_S} \operatorname{gradh}\Phi.d\mathbf{S} = 2\pi\int\sigma, \tag{41}$$

where $\int\sigma$ is the total strength of the sources enclosed by S (and given as a line, surface, or volume distribution). The same applies in the limit when the distribution is actually bounded in parts by S.

Equation (41) shows that the finite part of an infinite integral is more than an artificial analytical concept, and that in certain cases it may have a definite physical meaning. In fact, the product of density and c-flow, $-\rho\int\operatorname{gradh}\Phi.d\mathbf{S}$, is the linearized expression for the total flow of matter whenever that expression exists, for instance if Φ is given by a homogeneous volume distribution of sources over the interior of a small sphere of radius ϵ and centre P inside S, together with $-Ux$ corresponding to the free stream velocity. If σ is the total strength of the distribution, then

according to (41), $-\rho\int \operatorname{gradh}\Phi.d\mathbf{S} = 2\pi\sigma\rho$. Thus, $2\pi\sigma\rho$ is the rate at which matter is produced inside S. Now let ϵ tend to 0, while σ is kept constant. Then Φ tends to $-Ux+\Phi_P$, where Φ_P is the potential of a source of strength σ located at P. But σ having been kept constant, it follows that the rate at which matter is produced inside S, and hence the rate at which matter crosses S, is still $2\pi\sigma\rho$. And this, by (38), can be expressed by

$$-\rho\overset{*}{\int}\operatorname{gradh}\Phi_P.d\mathbf{S} = -\rho\overset{*}{\int}\operatorname{gradh}(-Ux+\Phi_P).d\mathbf{S},$$

so that the finite part is the natural generalization of an ordinary integral when the latter diverges.

Applying (41) to a small surface surrounding a point P inside a volume distribution of sources, we obtain

$$-\overset{*}{\int_S}\operatorname{gradh}\Phi.d\mathbf{S} = 2\pi\int_V \sigma\,dV,$$

and transforming the left-hand side by means of the divergence theorem, this becomes

$$-\overset{*}{\int_V}\operatorname{div}\operatorname{gradh}\Phi\,dV = 2\pi\int_V \sigma\,dV.$$

Since this is true for an arbitrary small volume containing P, we must have

$$\Delta\mathrm{h}\Phi = \operatorname{div}\operatorname{gradh}\Phi = -2\pi\sigma, \tag{42}$$

which is the counterpart of Poisson's theorem in subsonic theory.

Conversely, given the differential equation (42) over a certain region R, a particular solution of it is

$$\Phi = \int_R \sigma\Phi_P^*\,dV. \tag{43}$$

The general solution, as is easily seen by subtraction, then is

$$\Phi = \int_R \sigma\Phi_P^*\,dV+\overline{\Phi},$$

where $\overline{\Phi}$ is an arbitrary solution of (25).

Given a surface distribution of sources, it can be shown that the components of the gradient and therefore of the hyperbolic gradient of the potential remain finite and continuous on either side of the surface S. Also, Φ, and therefore its tangential derivatives, are continuous across the surface.

In order to find the discontinuity of the normal derivative across S, we apply (41) to a small cylinder whose bases are parallel to the surface on either side of it, and whose height is again small compared with its lateral

dimensions (Fig. 2). Letting first the height of the cylinder tend to 0 we find that

$$\int_{S'_+} \operatorname{gradh}\Phi . d\mathbf{S} - \int_{S'_-} \operatorname{gradh}\Phi . d\mathbf{S} = 2\pi \int_{S'} \sigma\, dS,$$

where S' denotes the portion of S inside the cylinder, and S'_+ and S'_- denote the two bases of the cylinder respectively, S'_+ being the base whose outside normal coincides with the outside normal of S. Letting S' tend to 0 round any given point on S, we obtain, denoting by λ, μ, ν the

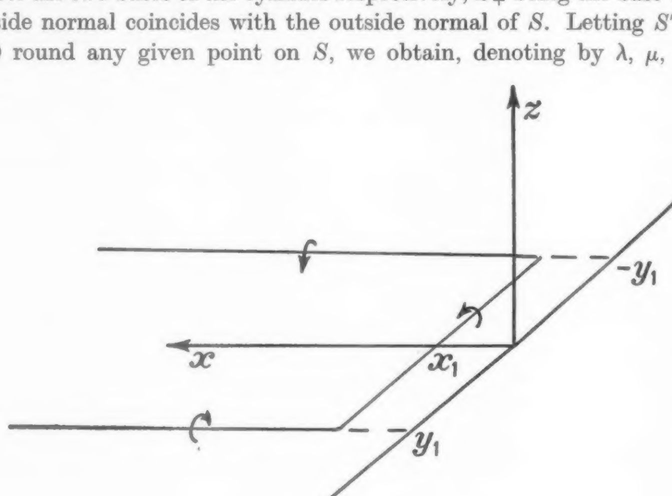

Fig. 2.

direction cosines of S at the point in question, and indicating by $\pm$ the derivatives of Φ on the two sides respectively,

$$-\lambda\beta^2\left(\frac{\partial\Phi}{\partial x_+}-\frac{\partial\Phi}{\partial x_-}\right)+\mu\left(\frac{\partial\Phi}{\partial y_+}-\frac{\partial\Phi}{\partial y_-}\right)+\nu\left(\frac{\partial\Phi}{\partial z_+}-\frac{\partial\Phi}{\partial z_-}\right) = 2\pi\sigma. \tag{44}$$

In particular, if the distribution is in the (x,y)-plane, we have $\lambda = \mu = 0$, $\nu = 1$, so that

$$\frac{\partial\Phi}{\partial z_+}-\frac{\partial\Phi}{\partial z_-} = 2\pi\sigma, \tag{45}$$

a result which is of considerable importance for the calculation of the wave drag of an aerofoil moving at supersonic speed at zero incidence (refs. **4**, **5**, **6**). A similar relation for the discontinuity of the potential across a doublet distribution in the (x,y)-plane, and which can be derived from (42), is fundamental in the supersonic theory of flat aerofoils at incidence.

These relations have hitherto been inferred by analogy with incompressible theory and then proved *ad hoc* in the particular cases required.

The need for a more systematic development was pointed out in the introduction to ref. **5**.

Dividing (44) by $\Delta = \sqrt{\{(\lambda\beta)^2+\mu^2+\nu^2\}}$ we obtain the result that the discontinuity of $\partial\Phi/\partial n'$ in a direction $\mathbf{n}'$ whose components (direction cosines) are $\left(-\frac{\lambda\beta^2}{\Delta}, \frac{\mu}{\Delta}, \frac{\nu}{\Delta}\right)$ is $\frac{2\pi\sigma}{\Delta}$. And since the tangential derivatives of Φ are continuous across the surface, it follows that the discontinuity of $\partial\Phi/\partial n'$ must be the discontinuity of $\partial\Phi/\partial n$, where $\mathbf{n} = (\lambda, \mu, \nu)$ is normal to S, multiplied by the cosine between $\mathbf{n}$ and $\mathbf{n}'$. Hence

$$\frac{\partial\Phi}{\partial n_+} - \frac{\partial\Phi}{\partial n_-} = 2\pi\sigma(-\lambda^2\beta^2+\mu^2+\nu^2)/\Delta$$

or

$$\frac{\partial\Phi}{\partial n_+} - \frac{\partial\Phi}{\partial n_-} = 2\pi \frac{1-\lambda^2(1+\beta^2)}{\sqrt{\{1-\lambda^2(1-\beta^4)\}}}\sigma. \tag{46}$$

Equation (46) amends the statement in ref. **5** that the discontinuity of the normal derivatives is always $2\pi\sigma$. The particular case in which this statement was applied, however, viz. (45), remains correct.

The chief use to which Hadamard puts his concept of the finite part is related to the above applications but is the outcome of a rather different approach. Hadamard's purpose is the solution of Cauchy's initial value problem for a very general class of hyperbolic partial differential equations including (20) as a special case. We are going to develop Hadamard's result in respect of equation (20), i.e. we are going to find an expression for the value of a solution Φ of (20) at a point $P = (x_0, y_0, z_0)$ inside a closed surface S, when the values of Φ and of $\partial\Phi/\partial n'$ are known on S, where, for every point of S, the direction $\mathbf{n}'$ is defined as above. An equivalent formula had been derived previously by Volterra (**8**).

Let Ψ_P^* be the velocity potential of a counter-source of unit strength located at P; then the function $\Phi\Psi_P^*$ is admissible inside S, excluding only P, provided Φ is regular inside S—and this can in fact be verified *a posteriori*.

Let us surround P by a small surface S' and apply equation (32) to the volume V bounded by $S+S'$. Since $\Delta\mathrm{h}\Phi = \Delta\mathrm{h}\Psi_P^* = 0$, we obtain

$$\overset{*}{\int\limits_{S+S'}} (\Phi\,\mathrm{gradh}\,\Psi_P^* - \Psi_P^*\,\mathrm{gradh}\,\Phi).d\mathbf{S} = 0,$$

or, taking the *inward* normal as the direction of a surface element of S, and the *outward* normal as the direction of a surface element of S',

$$\overset{*}{\int\limits_{S'}} (\Phi\,\mathrm{gradh}\,\Psi_P^* - \Psi_P^*\,\mathrm{gradh}\,\Phi).d\mathbf{S} = \overset{*}{\int\limits_{S}} (\Phi\,\mathrm{gradh}\,\Psi_P^* - \Psi_P^*\,\mathrm{gradh}\,\Phi).d\mathbf{S}. \tag{47}$$

It can be shown that as S' contracts to the point P, the left-hand side of (47) tends to $2\pi\Phi(x_0, y_0, z_0)$. Thus

$$2\pi\Phi(x_0, y_0, z_0) = \int\limits_S^{*} (\Phi\,\mathrm{gradh}\,\Psi_P^* - \Psi_P^*\,\mathrm{gradh}\,\Phi).d\mathbf{S}. \tag{48}$$

Denoting by λ, μ, ν the direction cosines of the normal to dS, we have, for any scalar function Φ,

$$\mathrm{gradh}\,\Phi.d\mathbf{S} = \left(-\lambda\beta^2\frac{\partial\Phi}{\partial x} + \mu\frac{\partial\Phi}{\partial y} + \nu\frac{\partial\Phi}{\partial z}\right)dS.$$

For $\beta = 1$, this becomes $\mathrm{gradh}\,\Phi.d\mathbf{S} = -\dfrac{\partial\Phi}{\partial n^*}\,dS$, where $\mathbf{n}^* = (\lambda, -\mu, -\nu)$.

Hence, for $\beta = 1$,

$$2\pi\Phi(x_0, y_0, z_0) = \int\limits_S^{*} \left(\Psi_P^*\frac{\partial\Phi}{\partial n^*} - \Phi\frac{\partial\Psi}{\partial n^*}\right)dS. \tag{49}$$

This is Hadamard's formula (58) (ref. **1**, p. 207) for the special case $f = 0$. The direction $\mathbf{n}^*$ is called by Hadamard the transversal direction to dS. Its geometrical interpretation, due to Coulon (compare ref. **1**), is that it is conjugate to the tangent plane to dS with respect to the cone $(x-x_1)^2-[(y-y_1)^2+(z-z_1)^2] = 0$, whose vertex is located at dS.

5. Vorticity distributions in steady supersonic flow

We now direct our attention to the study of rotational motion.

Given a field vector $\mathbf{q} = (u, v, w)$ we denote by ξ, η, ζ the components of $\mathrm{curl}\,\mathbf{q}$, $\xi = \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}$, $\eta = \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}$, $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$. The differential equation of the system of vortex lines is $dx/\xi = dy/\eta = dz/\zeta$ as usual, and the strength of a vortex tube is defined as the product of the cross-section σ into the resultant vorticity $\omega = (\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}}$, and is the same at all points of a vortex. All these results and definitions are in fact quite independent of whether the fluid is compressible or incompressible, except that in the case of supersonic flow it may be necessary to consider the finite parts of integrals of the type $\int\limits_C \mathbf{q}.d\mathbf{l}$ and $\int\limits_S \mathrm{curl}\,\mathbf{q}.d\mathbf{S}$ in cases where the ordinary integrals do not exist.

Applying the vector operator $\nabla\mathrm{h}$ in cross-multiplication to a vector $\mathbf{q} = (u, v, w)$, we obtain a vector which will be called $\mathrm{curlh}\,\mathbf{q}$ (hyperbolic curl of $\mathbf{q}$).

Explicitly—

$$\mathrm{curlh}\,\mathbf{q} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} + \beta^2\frac{\partial w}{\partial x}, -\beta^2\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{50}$$

Direct calculation shows that

$$\operatorname{divh}\operatorname{curlh}\mathbf{q} = 0 \tag{51}$$

and

$$\operatorname{curl}\operatorname{curlh}\mathbf{q} = \operatorname{gradh}\operatorname{div}\mathbf{q} - \operatorname{div}\operatorname{gradh}\mathbf{q}. \tag{52}$$

A field vector $\mathbf{q}$ will be called irrotational or lamellar, as usual, if $\operatorname{curl}\mathbf{q} = 0$, and it will be called hyperbolic solenoidal if $\operatorname{divh}\mathbf{q} = 0$.

We are going to show that a vector $\mathbf{q}$ defined in a region R and admissible in it can be represented as the sum of three vectors, one irrotational, one hyperbolic solenoidal, and one both irrotational and hyperbolic solenoidal. More precisely, we are going to represent $\mathbf{q}$ as $\mathbf{q} = \mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3$, where

$$\operatorname{divh}\mathbf{q}_1 = \operatorname{divh}\mathbf{q}, \quad \operatorname{curl}\mathbf{q}_1 = 0 \quad \text{in } R, \tag{53}$$

$$\operatorname{curl}\mathbf{q}_2 = \operatorname{curl}\mathbf{q}, \quad \operatorname{divh}\mathbf{q}_2 = 0 \quad \text{in } R, \tag{54}$$

and

$$\operatorname{divh}\mathbf{q}_3 = 0, \qquad \operatorname{curl}\mathbf{q}_3 = 0 \quad \text{in } R. \tag{55}$$

Assuming that vectors as described in (53) and (54) have been found, we put $\mathbf{q}_3 = \mathbf{q}-\mathbf{q}_1-\mathbf{q}_2$. Then $\operatorname{divh}\mathbf{q}_3 = \operatorname{divh}\mathbf{q}-\operatorname{divh}\mathbf{q}_1-\operatorname{divh}\mathbf{q}_2 = 0$, and $\operatorname{curl}\mathbf{q}_3 = \operatorname{curl}\mathbf{q}-\operatorname{curl}\mathbf{q}_1-\operatorname{curl}\mathbf{q}_2 = 0$, so that $\mathbf{q}_3$ defined in that way satisfies (55).

Putting $\sigma = \dfrac{1}{2\pi}\operatorname{divh}\mathbf{q}$ in R, we determine a scalar function Φ by $\Phi = \overset{*}{\int\limits_R} \sigma\Phi_P^*\,dV$, so that $\Delta\mathrm{h}\,\Phi = -2\pi\sigma$ according to (42), i.e. $\Delta\mathrm{h}\,\Phi = -\operatorname{divh}\mathbf{q}$, and so $\mathbf{q}_1 = -\operatorname{grad}\Phi$ satisfies (53). Thus

$$\mathbf{q}_1 = -\frac{1}{2\pi}\operatorname{grad}\overset{*}{\int\limits_R} \operatorname{divh}\mathbf{q}\,.\,\Phi_P^*\,dV. \tag{56}$$

To find $\mathbf{q}_2$, we shall assume that $\mathbf{q}_2$ is given as the hyperbolic curl of a vector $\mathbf{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$, $\mathbf{q}_2 = \operatorname{curlh}\mathbf{\Psi}$, and so by (52)

$$\operatorname{curl}\mathbf{q}_2 = \operatorname{curl}\operatorname{curlh}\mathbf{\Psi} = \operatorname{gradh}\operatorname{div}\mathbf{\Psi} - \operatorname{div}\operatorname{gradh}\mathbf{\Psi}. \tag{57}$$

We now restrict $\mathbf{\Psi}$ by the condition that $\operatorname{div}\mathbf{\Psi} = 0$. Then we must have $\operatorname{div}\operatorname{gradh}\mathbf{\Psi} = -\operatorname{curl}\mathbf{q}_2 = -\operatorname{curl}\mathbf{q}$, by (54) and (57), i.e.

$$\Delta\mathrm{h}\,\Psi_1 = -\xi, \quad \Delta\mathrm{h}\,\Psi_2 = -\eta, \quad \Delta\mathrm{h}\,\Psi_3 = -\zeta \quad \text{in } R, \tag{58}$$

and this according to (42) is solved by

$$\Psi_1 = \frac{1}{2\pi}\overset{*}{\int\limits_R} \xi\Phi_P^*\,dV, \qquad \Psi_2 = \frac{1}{2\pi}\overset{*}{\int\limits_R} \eta\Phi_P^*\,dV, \qquad \Psi_3 = \frac{1}{2\pi}\overset{*}{\int\limits_R} \zeta\Phi_P^*\,dV,$$

or

$$\mathbf{\Psi} = \frac{1}{2\pi}\overset{*}{\int\limits_R} \operatorname{curl}\mathbf{q}\Phi_P^*\,dV. \tag{59}$$

Then $\mathbf{q}_2 = \operatorname{curlh}\mathbf{\Psi}$ satisfies (54), provided we can show that in fact $\operatorname{div}\mathbf{\Psi} = 0$, as assumed. And this can be shown exactly as in the classical counterpart (ref. **7**, para. 148), provided the integrand vanishes at infinity or, in particular, if it vanishes outside a finite region. This again will certainly be the case if curl $\mathbf{q}$ vanishes for sufficiently small x, since Φ_P vanishes for sufficiently large x.

It is clear from the above construction that $\mathbf{q}_1$ determines the flow due to the source distribution in R, while $\mathbf{q}_2$ represents the flow due to the vorticity distribution. Given a three-dimensional vorticity distribution we then have in detail

$$\mathbf{\Psi}(x,y,z) = \frac{1}{2\pi}\overset{*}{\int\limits_{R'}}(\xi,\eta,\zeta)\frac{dx_0\,dy_0\,dz_0}{\sqrt{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]}} \tag{60}$$

(the expression on the right-hand side is actually an ordinary improper integral), where R' is the sub-domain of R which satisfies

$$(x-x_0)^2 > \beta^2[(y-y_0)^2+(z-z_0)^2] \quad \text{and} \quad x_0 < x.$$

We then obtain for the components u, v, w, of $\mathbf{q}_2 = \operatorname{curlh}\mathbf{\Psi}$,

$$\left.\begin{aligned}
u &= \frac{\beta^2}{2\pi}\overset{*}{\int\limits_{R'}}[(z-z_0)\eta-(y-y_0)\zeta]\frac{dx_0\,dy_0\,dz_0}{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]^{\frac{3}{2}}}\\
v &= \frac{\beta^2}{2\pi}\overset{*}{\int\limits_{R'}}[(x-x_0)\zeta-(z-z_0)\xi]\frac{dx_0\,dy_0\,dz_0}{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]^{\frac{3}{2}}}\\
w &= \frac{\beta^2}{2\pi}\overset{*}{\int\limits_{R'}}[(y-y_0)\xi-(x-x_0)\eta]\frac{dx_0\,dy_0\,dz_0}{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]^{\frac{3}{2}}}
\end{aligned}\right\}. \tag{61}$$

This may be written

$$\mathbf{q}_2 = -\overset{*}{\int\limits_{R'}}(\mathbf{r}\wedge\operatorname{curl}\mathbf{q})\frac{dV}{s^3}, \tag{62}$$

where $\mathbf{r} = (x-x_0, y-y_0, z-z_0)$, and $s = [(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]^{\frac{1}{2}}$. The corresponding formula for incompressible flow is

$$\mathbf{q}_2' = \overset{*}{\int\limits_{R'}}(\mathbf{r}\wedge\operatorname{curl}\mathbf{q}')\frac{dV}{r^3}. \tag{63}$$

The discrepancy in sign is only apparent, since as, for instance, formula (8) shows, the sign of a finite part does not follow the sign of the integrand, as for ordinary integrals.

We may now calculate the field of flow due to an isolated re-entrant

line vortex C. Replacing the volume element $dx_0\,dy_0\,dz_0$ in (60) by $\sigma_0\,dl_0$, where dl_0 is the element of length of C, and σ_0 its infinitesimal cross-section, and writing $\omega = (\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}}$, we have

$$\xi = \omega\,\frac{dx_0}{dl_0}, \qquad \eta = \omega\,\frac{dy_0}{dl_0}, \qquad \zeta = \omega\,\frac{dz_0}{dl_0},$$

and so, since $\omega\sigma_0$ is a constant, K, and since $d\mathbf{l}_0 = (dx_0, dy_0, dz_0)$, (60) becomes

$$\mathbf{\Psi}(x,y,z) = \frac{K}{2\pi}\int\limits_{C'}^{*} \frac{d\mathbf{l}_0}{\sqrt{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]}}, \tag{64}$$

where C' consists of the segments of C which satisfy

$$(x-x_0)^2 \geqslant \beta^2[(y-y_0)^2+(z-z_0)^2] \quad \text{and} \quad x > x_0.$$

If in particular C consists of straight segments which are parallel either to the x-axis or to the y-axis, then (64) can be integrated, since

$$\left.\begin{aligned} &\int \frac{dx_0}{\sqrt{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]}} \\ &\qquad\qquad = -\cosh^{-1}\frac{x-x_0}{\beta\sqrt{\{(y-y_0)^2+(z-z_0)^2\}}}+\text{const.} \\ &\text{and} \\ &\int \frac{dy_0}{\sqrt{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]}} \\ &\qquad\qquad = -\frac{1}{\beta}\sin^{-1}\frac{\beta(y-y_0)}{\sqrt{\{(x-x_0)^2-\beta^2(z-z_0)^2\}}}+\text{const.} \end{aligned}\right\}. \tag{65}$$

Now let C be a horseshoe vortex of strength K, consisting of the straight segments $(x_1 \leqslant x_0 < \infty,\ y_0 = -y_1,\ z_0 = 0)$, $(x_0 = x_1,\ -y_1 \leqslant y_0 \leqslant y_1,\ z_0 = 0)$ and $(x_1 \leqslant x_0 < \infty,\ y_0 = y_1,\ z_0 = 0)$, where x_1 and y_1 are given constants (Fig. 2). Using (65), we find that $\Psi_3 = 0$ always, and $\Psi_1 = \Psi_2 = 0$ for $x < x_1$, while for $x > x_1$,

$$\Psi_1 = \frac{K}{2\pi}\left[\cosh^{-1}\frac{x-x_1}{\beta\sqrt{\{(y-y_1)^2+z^2\}}}-\cosh^{-1}\frac{x-x_1}{\beta\sqrt{\{(y+y_1)^2+z^2\}}}\right], \tag{66}$$

where the $\cosh^{-1}$ are to be replaced by 0 when their arguments are smaller than 1, respectively, and

$$\Psi_2 = -\frac{K}{2\pi\beta}\left[\sin^{-1}\frac{\beta(y-y_1)}{\sqrt{\{(x-x_1)^2-\beta^2z^2\}}}-\sin^{-1}\frac{\beta(y+y_1)}{\sqrt{\{(x-x_1)^2-\beta^2z^2\}}}\right], \tag{67}$$

where the $\sin^{-1}$ are replaced by $+\frac{1}{2}\pi$ or $-\frac{1}{2}\pi$ when their arguments are greater than 1, or smaller than -1, respectively.

We now obtain $\mathbf{q}_2 = (u, v, w)$ by taking the hyperbolic curl of $\Psi = (\Psi_1, \Psi_2, 0)$, so that $u = v = w = 0$ for $x_1 > x$ while for $x > x_1$,

$$u = \frac{K\beta^2}{2\pi}\left[\frac{(y-y_1)z}{[(x-x_1)^2-\beta^2 z^2]\{(x-x_1)^2-\beta^2[(y-y_1)^2+z^2]\}^{\frac{1}{2}}} - \right.$$
$$\left. -\frac{(y+y_1)z}{[(x-x_1)^2-\beta^2 z^2]\{(x-x_1)^2-\beta^2[(y+y_1)^2+z^2]\}^{\frac{1}{2}}}\right],$$
$$v = \frac{K}{2\pi}\left[\frac{(x-x_1)z}{[(y-y_1)^2+z^2]\{(x-x_1)^2-\beta^2[(y-y_1)^2+z^2]\}^{\frac{1}{2}}} - \right.$$
$$\left. -\frac{(x-x_1)z}{[(y+y_1)^2+z^2]\{(x-x_1)^2-\beta^2[(y+y_1)^2+z^2]\}^{\frac{1}{2}}}\right],$$
$$w = \frac{K}{2\pi}\left[\frac{(x-x_1)(y-y_1)\{(x-x_1)^2-\beta^2[(y-y_1)^2+2z^2]\}}{[(x-x_1)^2-\beta^2 z^2][(y-y_1)^2+z^2]\{(x-x_1)^2-\beta^2[(y-y_1)^2+z^2]\}^{\frac{1}{2}}} - \right.$$
$$\left. -\frac{(x-x_1)(y+y_1)\{(x-x_1)^2-\beta^2[(y+y_1)^2+2z^2]\}}{[(x-x_1)^2-\beta^2 z^2][(y+y_1)^2+z^2]\{(x-x_1)^2-\beta^2[(y+y_1)^2+z^2]\}^{\frac{1}{2}}}\right], \tag{68}$$

where, for given x, y, z, the imaginary terms are omitted. Except for the notation, equations (68) agree with the field of flow round a horseshoe vortex calculated by Schlichting by an entirely different method (ref. **2**).

Some care is required when attempting to represent a volume or surface distribution of vortices as a combination of line vortices. Thus, according to (68), the components u, v, w, all vanish when (x, y, z) is outside both the cones $(x-x_1)^2-\beta^2[(y\pm y_1)^2+z^2]$ emanating from the tips. But it can be shown that this is no longer the case if the vorticity in the spanwise segment is distributed over a finite width Δx_0. However, even then the failure (which is due to the discontinuity of Ψ for line vortices) can occur only at points belonging to the envelope of the cones of type

$$(x-x_0)^2-\beta^2[(y-y_0)^2+(z-z_0)^2] = 0$$

emanating from the vortex lines which are supposed to generate the surface or volume.

6. Review of principal points

The present paper contains various developments of the analogy between linearized supersonic flow and classical incompressible hydrodynamics, with a view to applications in supersonic aerofoil theory.

The equation satisfied by the velocity potential in linearized supersonic flow is

$$-\beta^2\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2} = 0, \tag{69}$$

where $\beta^2 = M^2-1 = U^2/a^2-1$, $M = U/a$ being the Mach number, a the

velocity of sound, and U the magnitude of the free stream velocity. The direction of the free stream is supposed to be parallel to the positive direction of the x-axis.

Elementary solutions of equation (69) are the functions $\Phi_P(x,y,z)$ defined by

$$\left.\begin{aligned} \Phi_P(x,y,z) &= \frac{\sigma}{\sqrt{[(x-x_0)^2-\beta^2\{(y-y_0)^2+(z-z_0)^2\}]}} \\ &\quad \text{for } (x-x_0)^2 > \beta^2[(y-y_0)^2+(z-z_0)^2], \quad x > x_0 \\ \text{and}\quad \Phi_P(x,y,z) &= 0 \quad \text{elsewhere,} \end{aligned}\right\} \tag{70}$$

where $P = (x_0, y_0, z_0)$ and σ are arbitrary. Φ_P is said to be the velocity potential of a source of strength σ located at P. The actual velocity potential of a physical (acoustic) source travelling with a velocity $-U$, in a frame of reference travelling with the source is obtained by adding $-Ux$ to Φ_P.

The velocity potential of a supersonic doublet is, by definition, obtained by differentiating Φ_P with respect to length, relative to the coordinates of P in any given direction. As in incompressible flow, a supersonic doublet can be interpreted as a combination of two supersonic sources whose distance δ from each other tends to 0, and whose strengths are of equal magnitude and of opposite sign, $\pm\sigma'$, and related to δ in such a way that the product $\sigma'\delta$ remains finite as δ tends to 0.

The importance of these and other particular solutions of (69) mentioned in §§ 4 and 5 above depends not so much on their direct physical meaning as on the fact that the flow round aerofoils and other bodies under supersonic conditions can be built up from these solutions. With a view to such applications, general line, surface, and volume distributions of sources and doublets are considered in § 4, and similar distributions of vorticity in § 5. In § 4 results are derived corresponding to the theorems of Gauss (on total normal intensity) and of Poisson in classical potential theory. The discontinuity of the normal component of velocity across a surface distribution of sources is calculated as an application.

The velocity field due to an arbitrary distribution of vorticity is calculated in § 5. In particular, formulae are derived for the field of flow due to an isolated re-entrant vortex travelling at a steady speed. These formulae constitute a supersonic counterpart to the law of Biot-Savart. Finally, it is shown that for the case of a horseshoe vortex the field of flow obtained by the method of the present paper agrees with that calculated by Schlichting in an entirely different manner. In contrast with the fictitious character of sources and sinks in supersonic aerofoil theory, the occurrence of trailing vortices in the wake of an aerofoil is

a matter of physical reality. On the other hand, the vorticity induced by a shock wave is not taken into account by linearized theory.

Two mathematical ideas are required to carry out the above analysis. One is the purely formal introduction of the vector operator $\nabla h \beta$ ('hyperbolic nabla of index β') defined by $\left(-\beta^2 \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, which is used in conjunction with the ordinary vector operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. The hyperbolic nabla, like the ordinary nabla, can be applied either to a scalar, or, in scalar and in vector multiplication, to a vector.

The second concept required for the analysis of the present paper is that of the finite part of an infinite integral. It was introduced by Hadamard in connexion with the solution of Cauchy's problem in partial differential equations (ref. **1**). This concept enables us to cope with the infinite integrals that occur in the analysis of supersonic flow, e.g. in calculating the flow across a surface surrounding an isolated supersonic source. For the purposes of the present paper, the concept had to be developed in rather greater generality than considered by Hadamard. Also, in addition to Green's theorem for finite parts, a particular case of which is proved and applied by Hadamard, we require a proof of the extension of Stokes' theorem to include finite parts.

The analysis of the present paper has been applied to the calculation of the downwash in the wake of a Delta wing. The results of this investigation are given elsewhere (ref. **9**).

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THE APPLICATION OF THE NATIONAL ACCOUNTING MACHINE TO THE SOLUTION OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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SUMMARY

Milne's formula for approximate quadrature is used as the basis of a method for the solution of first-order differential equations on the National machine. The method, which is illustrated by a numerical example, enables the machine to form the required dependent variable without the necessity for conversion from a sum to an integral.

1. Introduction

THE National Accounting Machine is now in fairly general use for scientific computing; it has been described in detail by Dr. L. J. Comrie (**1**), to whom belongs the credit for realizing the possibilities of the machine and for exploiting them. For those to whom the machine is unfamiliar, a very brief description of its principal features is given in the next section.

The machine has for long been applied to assist in the solution of differential equations of both first and second orders, but its main use has generally been restricted to saving the operator the labour of writing and of differencing and integration. Since the machine has a limited storage capacity and there is no means of direct multiplication, there are obvious difficulties in using the machine to convert derivatives to the differences which are necessary for building up the solution. A method for the solution of linear second-order differential equations, which overcomes this difficulty, has already been described (**2**); this method relies, however, on a mathematical transformation of the original variables.

The solution of first-order equations suffers from the disadvantage that first derivatives become available at the tabular interval, while the first differences are required at the intermediate half-way point. The method given below, in its simplest form, relieves the operator of all stages other than the calculation of the first derivative, which cannot be avoided.

2. The National machine

The National is a printing adding machine with six adding mechanisms, or registers, each with 12-figure capacity; two of these registers (Nos. 1 and 3) can subtract directly as well as add, but subtraction in the other

four must be performed by the addition of complements. A number set on the 12-bank keyboard can be added (including subtraction in registers 1 and 3) to any combination of the six registers; the contents of any selected register can be transferred, with (a total or T operation) or without (a sub-total or ST operation) clearing of the register selected; in each case the number added or transferred is printed. The destination of such a number is controlled by the 'stop' on which the machine is operated, and thus, in any particular set-up, by the position of the moving carriage; register selection can be made by the operator independently of the carriage position, but is in fact almost invariably governed by position.

The moving carriage carries a removable 'form-bar' on which stops can be easily placed in any desired position and order; it normally 'tabulates', or moves from one stop to the next, after each operation, automatically feeding the paper and returning to the first position, indicated by an M (margin) stop, after reaching the end of a line, indicated by an R (return carriage) stop.

It will suffice here to mention further four points of detail; stops can be made available to add in any combination of registers and, in particular, 'reversible' stops are available for registers 1 and 3 enabling them to be changed quickly from + to − and vice versa; the machine can be made to interpret the instruction to subtract the contents of a register from itself as an instruction to clear the register concerned; printing can be suppressed in any position by using an NP (non-print) stop; an R stop can, when necessary, be rendered ineffective to allow of the carriage moving to positions beyond it.

3. Notation and basic formulae

The differential equation will be written in the form:

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

The interval of tabulation is denoted by h, but, as is usual, consecutive values of the independent variable will be indicated by integers and corresponding values of y and f by suffixes. Central difference notation is used.

The fundamental requirement is for an equation connecting two, not necessarily consecutive, values of y with the derivatives, and their differences, at intermediate points. If the coefficients of the derivatives and differences can be reduced to small integers it may be possible to perform the simple multiplications and additions on the machine.

The following formulae are available:

$$\begin{aligned} y_n - y_{n-2} &= 2h(f + \tfrac{1}{6}\delta^2 - \tfrac{1}{180}\delta^4 + \ldots)_{n-1} \\ &= \tfrac{1}{3}h(f_{n-2} + 4f_{n-1} + f_n) - \tfrac{1}{90}h\delta^4 + \ldots \quad \text{(Simpson's rule)}; \end{aligned} \tag{2}$$

$$\begin{aligned} y_n - y_{n-4} &= 4h(f + \tfrac{2}{3}\delta^2 + \tfrac{7}{90}\delta^4 - \tfrac{2}{945}\delta^6 + \ldots)_{n-2} \\ &= \tfrac{4}{3}h(2f_{n-3} - f_{n-2} + 2f_{n-1}) + \tfrac{14}{45}h\delta^4 - \ldots \quad \text{(Milne's formula)}; \end{aligned} \tag{3}$$

$$y_n - y_{n-6} = 6h(f + \tfrac{3}{2}\delta^2 + \tfrac{11}{20}\delta^4 + \tfrac{41}{840}\delta^6 - \tfrac{3}{2800}\delta^8 + \ldots)_{n-3}. \tag{4}$$

The formulae with an odd number of intervals are generally unsuitable. Of those given (3) is by far the most tractable. Putting

$$g(x, y) = \tfrac{4}{3}hf(x, y), \tag{5}$$

$$y_n - y_{n-4} = (3g + 2\delta^2 + \tfrac{7}{30}\delta^4 - \tfrac{2}{315}\delta^6 + \ldots)_{n-2}, \tag{6}$$

where the differences now refer to g. The formula is used in this form rather than in the Lagrangian form for a number of reasons. Firstly, the National machine lends itself more readily to a formula containing differences instead of a series of ordinates, secondly, differences give some form of check on the integration, and thirdly, the fourth differences are required in order to allow for their effect.

4. National set-up

The principle of the set-up is to difference g_n to the fourth difference, at the same time building up $(3g + 2\delta^2)_{n-1}$ in an empty register; the addition of y_{n-3} will then give y_{n+1}, which is used to calculate g_{n+1} and thus to recommence the cycle. Provision is made for applying the fourth difference correction without having to reset the machine.

The set-up is given in detail in Fig. 1; much of the complication arises from the necessity to cater for varying signs. In the following description such details will be ignored; these can be followed from the set-up by those possessing a National machine.

In addition to the National, a calculating machine is required for computing g from a knowledge of x and y, and a second machine or table for finding $\tfrac{7}{30}\delta^4$.

The normal cycle consists of:

(i) Operate the machine in positions 1, 2 (printing the argument and corresponding *approximate* function), 3, 4 and 5.

(ii) Calculate g_n from the value of y_n printed in position 2; in doing this arrange, if possible, that y is the last multiplier or that there is some simple method of correcting g differentially for a small change in y.

(iii) Set this approximate value of g_n in position 6 and operate the machine in position 7 to print the corresponding δ^4_{n-2}.

FIG. 1. NATIONAL SET-UP FOR THE INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Position	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Stop	M	0	0	±3 [NP]	±3 [NP]	±3,-1	±1	-1	6	5,-1	±3,2	4,-1	±3,-1 [NP]	±3	±3	0,R	4	R
Operation	-	Set	T3	ST4	ST4	ST4	Set	ST1	Set	ST6	ST5	ST2	ST5	Set	Set	-	-	-
Print	-	Arg.x = nh	$\pm y_n$	$[g_{n-1}]$	$[g_{n-1}]$	g_{n-1}	$\pm g_n$ (approx)	δ^4_{n-2} (approx)	δ^4_{n-2}	$\delta^3_{n-\frac{3}{2}}$	δ^2_{n-1}	$\delta^1_{n-\frac{1}{2}}$	$[\delta^2_{n-1}]$	$\pm y_{n-3}$	$\pm\frac{7}{30}\delta^4_{n-1}$ (extrapolated)	-	-	-
Initial Set-up	Line 1:	Set x_1	Set y_1	-	-	-	Set g_1	-	T1	ST6	ST5	ST2	ST5	-	-	-	Line 5 will be a normal operation	
	2:	Set x_2	Set y_2	ST4	ST4	ST4	Set g_2	-	T1	ST6	ST5	ST2	ST5	-	-	-		
	3:	Set x_3	Set y_3	ST4	ST4	ST4	Set g_3	-	T1	ST6	ST5	ST2	ST5	-	-	T3		
	4:	Set x_4	Set y_4	ST4	ST4	ST4	Set g_4	-	T1	ST6	ST5	ST2	ST5	Set y_1	Set $\frac{7}{30}\delta^4_3$	-		
Set-up from completed line	-	-	-	-	-	-	-	-	Set $\delta^3_{n-\frac{3}{2}}$ with S1 key down	Set δ^2_{n-1}	Set $\delta^1_{n-\frac{1}{2}}$	ST2	-	T3	Set $\pm y_{n+1}$ (from next line) on +3	-	Set g_{n-1}	-

Normal operation will produce the next line (i.e. line n + 1).

SIGNS OF STOPS

Position	3	4	5	6	10	12	13	14
Stop	±3	±3	±3,-1	±1	±3,2	±3,-1	±3	±3
Sign as	y_n	y_n	y_n	g_n	y_n	y_n	$y_n \times y_{n-3}$	$y_n \times \frac{7}{30}\delta^4_{n-1}$

CHANGES OF SIGN

y_n : on the last line before the change reverse the signs of the stops in positions 3,4,5,10,12,13,14.

g_n : reverse the ±1 stop in position 6.

y_{n-3} : reverse the ±3 stop in position 13.

$\frac{7}{30}\delta^4_{n-1}$: reverse the ±3 stop in position 14.

FORMULA $y_n = y_{n-4} + (3g + 2\delta^2 + \frac{7}{30}\delta^4 - \frac{2}{315}\delta^6 + \ldots)_{n-2}$, where $g = \frac{4}{3}hf(x,y)$ and the differences refer to g.

INITIAL PREPARATION Calculate the first five values of y and hence g; difference g to δ^4 and calculate $\frac{7}{30}\delta^4$.

DETAILED OPERATION (for a normal cycle)

(i) Operate the machine in positions 1 to 5 and check that g_{n-1} in position 5 is correct.
(ii) Calculate g_n (approx.) and set in position 6.
(iii) ST1 giving δ^4_{n-2} (approx.) in position 7.
(iv) Calculate $\frac{7}{30}\delta^4_{n-2}$ and compare with the extrapolated value at the end of the previous line.
(v) Adjust y_n in the calculation of g_n by the change in $\frac{7}{30}\delta^4_{n-2}$. Adjust δ^4_{n-2} by any change in g_n; this may again change $\frac{7}{30}\delta^4_{n-2}$ but a balance is soon achieved.
(vi) Correct y_n (position 2) and $\frac{7}{30}\delta^4_{n-2}$ (position 14, previous line) by hand.
(vii) Set the final δ^4_{n-2} in position 8 and operate the machine in positions 9 to 12.
(viii) Set the corrected y_{n-3} (from position 2, three lines back) in position 13.
(ix) Extrapolate δ^4_{n-1}, calculate $\frac{7}{30}\delta^4_{n-1}$ and set in position 14.

(iv) Using the table, or second machine, calculate $\frac{7}{30}\delta^4_{n-2}$ and use the difference between this and the extrapolated value (appearing in position 14 on the previous line) to correct the approximate value of y_n; change g_n accordingly, and δ^4_{n-2} *by the same amount*; if this change is large, it may change $\frac{7}{30}\delta^4_{n-2}$, y_n, g_n and δ^4_{n-2} itself; but the process is rapidly convergent and a balance is soon achieved. In practice, the interval h will be chosen so that the extrapolation can be done with little error. The final correction to y_n must be recorded, by hand or otherwise, either directly or in the form of an alteration of y_n, and the final δ^4_{n-2} entered in position 8.

(v) Operate the machine in positions 9, 10, 11, 12; these, in conjunction with the operations in positions 3, 4 and 5, result in putting $3g+2\delta^2$ into register 3, as well as completing the differencing cycle.

(vi) Set y_{n-3} from the *corrected* value in position 2 on line $n-3$.

(vii) Extrapolate δ^4_{n-1} and set $\frac{7}{30}\delta^4_{n-1}$ in position 14.

At convenient intervals, say, every 10 or 20 lines, the accuracy of the various settings should be checked and δ^4 should be differenced for checking purposes; a check on the accuracy of the machine is provided by comparing g_{n-1} printed in position 5 with the value printed in position 6 on the line above, corrected for the difference between the approximate and final values of δ^4_{n-3}. The set-up suffers from the disadvantage that y_n is checked only through the differencing of g_n; therefore the contributions to y_n set into the machine in positions 13 and 14 should be checked, since small errors might otherwise be undetected.

It might be thought that this cycle, requiring operation of the National machine, an ordinary calculating machine, reference to tables and some mental arithmetic, would impose a considerable strain on the computer. Actually, all the operations other than the extrapolation of δ^4 and the subsequent correction of y soon become almost automatic and the computer can concentrate upon the general flow of the solution, thus achieving a much higher accuracy in extrapolation than would be the case if his attention had to be diverted by writing or by mental additions and subtractions.

5. Accuracy of the method

The method can be made as accurate as desired by retaining sufficient figures and, if necessary, decreasing the interval. Although no provision is made in the set-up, it is not difficult to allow for the sixth difference correction by extrapolating it and setting after position 14; but if δ^6 is large enough for $\frac{2}{315}\delta^6$ to be sensible, extrapolation of δ^4 will have become difficult and either a change of interval or a reduction in the number of figures retained would be called for.

The building-up error, which cannot be avoided in numerical integration, is here quite different from that normally occurring. There are really four separate series, of which a typical one contains the sum of errors of the form:

$$2\epsilon_{n-3}+\epsilon_{n-2}+2\epsilon_{n-1}+\eta_{n-2}, \tag{7}$$

where the ϵ's and η's are elementary rounding-off errors. This compares with the straight sum of four rounding-off errors, which is the optimum that can be achieved for four steps. The building-up error is therefore not likely to be serious and the capacity of the National will allow ample guarding figures to be retained. The errors of the different series are not independent so that they cannot be determined by examining the differences; the second-order effect of these errors on the solution is beyond the scope of this paper.

6. Example

To illustrate the principles of the method it will be applied to the equation

$$\frac{dy}{dx} = x+y \tag{8}$$

with initial condition $y = 1$ at $x = 0$. This example has been deliberately chosen because of its simplicity and the constancy of sign of the differences. The interval h will be taken as 0·1, so that

$$g = 0{\cdot}133...x+0{\cdot}133...y. \tag{9}$$

Nine decimals will be retained.

By expansion in ascending series, by successive approximation, or by other methods (e.g. repeated use of Taylor's theorem) we find:

Line	*x*	*y*	*g and differences (unit 9th decimal)*				
−2	−0·2	+0·83746 1506	+ 8499 4867				
				+2296 1778			
−1	−0·1	0·90967 4836	10795 6645		+241 4910		
				2537 6688		+25 3980	
0	0·0	1·00000 0000	13333 3333		266 8890		+2 6710
				2804 5578		28 0690	
1	+0·1	1·11034 1836	16137 8911		294 9580		
				3099 5158			
2	+0·2	1·24280 5516	19237 4069				

As a check it is noted that

$$(3g+2\delta^2+\tfrac{7}{30}\delta^4)_0 = +0{\cdot}40534\ 4011,$$

whereas

$$y_2-y_{-2} = +0{\cdot}40534\ 4010.$$

These initial values are fed into the National according to the rules given on the set-up and the computer is faced with the figures appearing above the line in Fig. 2. Note that the value at 0·2 differs from the

Fig. 2. ILLUSTRATIVE EXAMPLE PHOTOGRAPHICALLY REPRODUCED (REDUCED 3:2) FROM FIGURES PRINTED BY THE NATIONAL MACHINE

1	2	5	6	7	8	9	10	11	13	14
n=10x	$y_n \times 10^9$	g_{n-1}	g_n(approx.)	δ^4_{n-2}(approx.)	δ^4_{n-2}	$\delta^3_{n-\frac{3}{2}}$	δ^2_{n-1}	$\delta^1_{n-\frac{1}{2}}$	y_{n-3}	$\frac{7}{30}\delta^4_{n-1}$ (extrapolated)
−2	837 461 506		84 994 867							
−1	909 674 836	84 994 867	107 956 645					22 961 778		
0	1 000 000 000	107 956 645	133 333 333				2 414 910	25 376 688		
1	1 110 341 836	133 333 333	161 378 911			253 980	2 668 890	28 045 578	837 461 506	6 232
2	1 242 605 517	161 378 911	192 374 069		26 710	280 690	2 949 580	30 995 158	909 674 836	$\overset{6\,888}{\cancel{7\,000}}$
3	1 399 717 $\overset{617}{\cancel{729}}$	192 374 069	226 629 031	29 534	29 519	310 209	3 259 789	34 254 947	1 000 000 000	7 6$\overset{12}{\cancel{00}}$
4	1 583 649 3$\overset{97}{\cancel{65}}$	226 629 016	264 486 585	32 624	32 625	342 834	3 602 623	37 857 570	1 110 341 836	8 4$\overset{14}{\cancel{00}}$
5	1 797 442 5$\overset{44}{\cancel{30}}$	264 486 586	306 325 671	36 058	36 060	378 894	3 981 517	41 839 087	1 242 805 517	9 $\overset{297}{\cancel{000}}$
6	2 044 237 60$\overset{6}{\cancel{9}}$	306 325 673	352 565 015	39 844	39 843	418 737	4 400 254	46 239 341	1 399 717 617	10 27$\overset{7}{\cancel{0}}$
7	2 327 505 4$\overset{21}{\cancel{14}}$	352 565 014	403 667 389	44 043	44 043	462 780	4 863 034	51 102 375	1 583 649 397	11 35$\overset{6}{\cancel{0}}$
8	2 651 081 8$\overset{63}{\cancel{57}}$	403 667 389	460 144 248	48 670	48 670	511 450	5 374 484	56 476 859	1 797 442 544	12 5$\overset{51}{\cancel{40}}$
9	3 019 206 2$\overset{30}{\cancel{19}}$	460 144 248	522 560 829	53 788	53 790	565 240	5 939 724	62 416 583	2 044 237 606	13 870
10	3 436 563 668	522 560 831	591 541 822	59 444						

computed value. The next stage is to extrapolate $\frac{7}{30}\delta_1^4$ and the computer, very much in the dark, chooses the convenient round number, 7000, which is set in position 14.

Stage (i) is then completed and the computer turns to the calculating machine to form g_3 from the recently printed y_3. Here $0{\cdot}133...x$ is put into the product register and 0·133..., on the keyboard, is multiplied by the printed value of y_3 to give an approximation to g_3, which is set in position 6. The next step is to print the approximate fourth difference δ_1^4, namely, 2 9534; this shows that 7000 was a very poor extrapolation for $\frac{7}{30}\delta_1^4$, the true value being 6891. This change enters directly into y_3 and therefore into the multiplier on the calculating machine, which is now reduced by $109 = 7000-6891$; this changes g_3 by 15, δ_1^4 by the same amount, and thus $\frac{7}{30}\delta_1^4$ and y_3 by 3; g_3 is, however, unchanged and so is $\frac{7}{30}\delta_1^4$. The whole process is shown in tabular form below:

	M.R. $= y_3$	P.R. $= g_3$	δ_1^4	$\frac{7}{30}\delta_1^4$
Extrapolated value				7000
First approx.	1·39971 7729	22662 9031	2 9534	6891
Second approx.	7620	9016	2 9519	6888
Final values	7617	9016		

The value of y_3 is now amended by hand (or the correction to $\frac{7}{30}\delta_1^4$ and y_3 printed by the machine) and the final value of $\delta_1^4 = 2\ 9519$ is set in position 8; there is no need to correct g_3 since the correct value will be built up from δ_1^4. Routine operation then brings the machine to position 13 in which the corrected value of y_0 (here 1·00000 0000) is carefully set. In the next position $\frac{7}{30}\delta_2^4$ must be extrapolated from the two printed fourth differences; a value of 7600 is chosen and the cycle recurs.

It will be noticed that, after the first, the extrapolated values are all within 14 of the final values, leading to a negligible second-order effect.

After six more cycles we obtain:

$$\text{at } x = 1{\cdot}0, \quad y_{10} = 3{\cdot}43656\ 3668 \quad \text{as compared with the}$$

$$\text{true value} \quad = 3{\cdot}43656\ 3657.$$

Now it will have been clear that sixth differences, although significant, have been ignored; since their effect is very small, the correction need not be applied to, say, y_3 until this value is required on line 6, by which time δ_1^6 is accurately known. The application of the two relevant corrections

$$-\tfrac{2}{315}\delta_4^6 = -3 \quad \text{and} \quad -\tfrac{2}{315}\delta_8^6 = -4$$

reduces the discrepancy in the check value. But even when allowance is made for the error of -1 in y_2 there is still a difference of 3, which is

rather large; examination shows that this is due to an unfortunate series of rounding-offs in g.

Detailed modifications are clearly possible; in particular the final correction to y can be set on and printed by the machine instead of altering the end figures of y by hand.

7. Acknowledgements

This application of the National machine arose out of a discussion with Professor W. G. Bickley as to the possibility of mechanizing the 'Milne-Simpson' method (using Milne's formula to extrapolate and Simpson's rule to check); his interest and assistance are greatly appreciated.

The paper is published with the permission of the Astronomer Royal.

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ON THE HODOGRAPH TRANSFORMATION FOR HIGH-SPEED FLOW

II. A FLOW WITH CIRCULATION

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SUMMARY

A steady two-dimensional isentropic compressible fluid flow about a contour with circulation, reducing to the potential flow with circulation about a circle as the Mach number at infinity tends to zero, is specified in the hodograph plane.

1. Introduction

In this paper the method of Part I (**1**) is applied to obtain the high-speed flow corresponding to the low-speed flow with circulation about a circular cylinder. The reasons for presenting this alternative treatment of the problem to those of refs. (**2**) and (**3**) are again those attaching to any such multiple treatment: it widens the basis of understanding and future development. The work of this part is necessarily more complicated than that of Part I, however; nor has it the generality and directness in argument of ref. (**2**). Its mathematical interest, in the author's belief, outweighs these objections.

2. The low-speed flow

If the circulation is $4\pi\sin\alpha$ and the velocity at infinity unity, the complex potential and complex velocity for the flow of 'perfect' fluid past the unit circle are

$$w = z+\frac{1}{z}+2i\sin\alpha\log z, \qquad \zeta = \frac{dw}{dz} = 1-\frac{1}{z^2}+\frac{2i\sin\alpha}{z}. \tag{1}$$

Inverting the latter equation as $z = (1-\zeta)^{-1}(-i\sin\alpha+(\cos^2\alpha-\zeta)^{\frac{1}{2}})$, we obtain for w the expression

$$w = \frac{1}{1-\zeta}(-i\sin\alpha+(\cos^2\alpha-\zeta)^{\frac{1}{2}})+i\sin\alpha+(\cos^2\alpha-\zeta)^{\frac{1}{2}}- \\ -2i\sin\alpha\log(i\sin\alpha+(\cos^2\alpha-\zeta)^{\frac{1}{2}}) = P(\zeta)+Q(\zeta), \tag{2}$$

where
$$P(\zeta) = i\sin\alpha\left(1-\frac{1}{1-\zeta}-\log(1-\zeta)\right), \tag{3}$$

and $Q(\zeta)$ can be put into the form

$$Q(\zeta) = \left(1+\frac{1}{1-\zeta}\right)(1-\zeta-\sin^2\alpha)^{\frac{1}{2}}+2\sin\alpha\sin^{-1}((1-\zeta)^{-\frac{1}{2}}\sin\alpha). \tag{4}$$

By expansion of Q in descending powers of $(1-\zeta)$, which are in turn expanded in ascending powers of ζ, and by inversion of this double sum, we find that the branch of $Q(\zeta)$ equal to $2(\cos\alpha+\alpha\sin\alpha)$ when $\zeta = 0$ has for its Taylor series near $\zeta = 0$ the expression

$$\sum_{n=0}^{\infty} \frac{\zeta^n}{n!} \frac{(n-1)(n-\frac{3}{2})!}{(-\frac{1}{2})!} F(n-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha). \tag{5}$$

The hypergeometric function appearing here is an integral function of n: it has the integral representation (for all n)

$$(\cos\alpha)^{1-2n} - (n-\tfrac{1}{2})\sin\alpha \int_0^{\sin^2\alpha} x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}-n}\,dx, \tag{6}$$

which shows that it is $O(|n(\cos^2\alpha)^{-n}|)$ as $|n| \to \infty$ for $\Re(n) > 0$.

Hence the series (5) converges for $|\zeta| < \cos^2\alpha$. When $\Re(n) < 0$ expression (6) becomes

$$-(n-\tfrac{1}{2})\sin\alpha \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}-n}\,dx + O(|n(\cos^2\alpha)^{-n}|)$$

$$= \frac{(-\frac{1}{2})!\,(-n+\frac{1}{2})!}{(-n)!}\sin\alpha + O(|n(\cos^2\alpha)^{-n}|) = O(|n|). \tag{7}$$

The series for $Q(\zeta)$ can be continued in the manner adopted in § 2 of Part I. In fact the series is

$$\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{(\nu-1)(-\nu-1)!\,(-\nu-\frac{3}{2})!}{(-\frac{1}{2})!} F(\nu-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha)(-\zeta)^\nu\,d\nu \tag{8}$$

within its circle of convergence $|\zeta| < \cos^2\alpha$, being minus the sum of the residues of the integrand at its poles $\nu = n \geqslant 0$ to the right of the contour; while for $|\zeta| > 1$ the integral, by (7), is equal to the sum of the residues of the integrand at its poles to the left of the contour, i.e. to

$$\sum_{n=0}^{\infty} \frac{(-n-\frac{1}{2})(n-\frac{3}{2})!}{n!\,(-\frac{1}{2})!} F(-n, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha)(-1)^n(-\zeta)^{\frac{1}{2}-n}. \tag{9}$$

Here, as in Part I, if $\arg\zeta > 0$, $\arg(-\zeta)$ must be taken as $\arg\zeta-\pi$ in order to secure the convergence of the integral (8), so that $(-\zeta)^{\frac{1}{2}-n}$ is $-i(-1)^n\zeta^{\frac{1}{2}-n}$; while if $\arg\zeta < 0$ then $\arg(-\zeta)$ is $\arg\zeta+\pi$ and $(-\zeta)^{\frac{1}{2}-n}$ is $i(-1)^n\zeta^{\frac{1}{2}-n}$. Thus, for $|\zeta| > 1$, if $\pm$ signifies the sign of $\arg\zeta$ on the path of continuation,

$$Q(\zeta) = \mp i \sum_{n=0}^{\infty} \frac{(-n-\frac{1}{2})(n-\frac{3}{2})!}{n!\,(-\frac{1}{2})!} F(-n, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha)\zeta^{\frac{1}{2}-n}. \tag{10}$$

This can be continued into the region $\cos^2\alpha < |\zeta| < 1$ by use of (7), whence

$$Q(\zeta) = \mp i \sum_{n=0}^{\infty} \frac{(-n-\frac{1}{2})(n-\frac{3}{2})!}{n!\,(-\frac{1}{2})!}\left[\frac{(-\frac{1}{2})!\,n!}{(n-\frac{1}{2})!}\sin\alpha + O(n\cos^{2n}\alpha)\right]\zeta^{\frac{1}{2}-n}$$

$$= \mp i \sum_{n=0}^{\infty} \frac{\frac{1}{2}+n}{\frac{1}{2}-n}\sin\alpha\,\zeta^{\frac{1}{2}-n} \tag{11}$$

plus a function analytic for $|\zeta| > \cos^2\alpha$; (11) in turn differs by another function analytic for $|\zeta| > \cos^2\alpha$ from

$$\mp i\sin\alpha[(1-\zeta)^{-1}+\log(1-\zeta)], \tag{12}$$

with which equation (3) should be compared. It is deduced that $P(\zeta)+Q(\zeta)$, continued analytically via the region $\arg\zeta < 0$, has no singularity at $\zeta = 1$; but that if continued via the region $\arg\zeta > 0$, it has a singularity there, that of $-2i\sin\alpha[(1-\zeta)^{-1}+\log(1-\zeta)]$. Also, by (10), $Q(\zeta)$ changes sign in going round a cut joining $\cos^2\alpha$ and 1. The Riemann surface corresponding to that in § 2 of Part I can now be depicted as below, Fig. 1. The values of w shown are those obtained by continuation

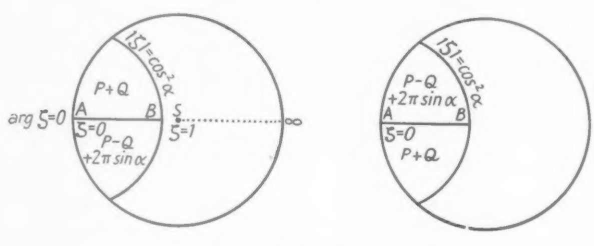

FIG. 1.

along paths not crossing a cut from infinity to the singularity S: w is one-valued on the surface so obtained. AB is the cut joining the two sheets. A positive encirclement of S adds $4\pi\sin\alpha$ to w. The only singularities of w are S and B, corresponding in the physical plane to the point at infinity and a certain finite point C where $z = -i\,\mathrm{cosec}\,\alpha$. The cut AB corresponds to a circle of diameter OC (O the origin) which passes through the two stagnation points (corresponding to A): on this circle $\theta = 0$.

3. The high-speed flow with $f_n(\tau_1) = e^{-ns_1}$

To obtain from this a high-speed flow, we write, as in Part I, § 3, corresponding to the above P, Q,

$$P = -i\sin\alpha\sum_{n=1}^{\infty}(1-n^{-1})\psi_n(\tau)e^{-n(s_1+i\theta)}, \tag{13}$$

$$Q = \sum_{n=0}^{\infty}\frac{(n-1)(n-\frac{3}{2})!}{n!\,(-\frac{1}{2})!}F(n-\tfrac{1}{2},-\tfrac{1}{2};\tfrac{1}{2};\sin^2\alpha)\psi_n(\tau)e^{-n(s_1+i\theta)}, \tag{14}$$

and consider $\Im(P+Q)$ as a commencing branch of ψ. The former series converges for $\tau < \tau_1$, the latter (by equation (6) and the asymptotic formula for $\psi_n(\tau)$) for $e^s < e^{s_1}\cos^2\theta$: since s is an increasing function of τ, this can be written $\tau < \tau_2$ for some τ_2.

When $\tau < \tau_2$ we have as in (8)

$$Q = \frac{1}{2\pi i}\int\limits_{-\infty i}^{\infty i} \frac{(\nu-1)(-\nu-1)!\,(\nu-\frac{1}{2})!}{(-\frac{1}{2})!} F(\nu-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha)\psi_\nu(\tau)e^{\nu(\mp i\pi-s_1-i\theta)}\,d\nu, \tag{15}$$

with the $\mp$ sign according as $\theta \lesseqgtr 0$. For $\tau > \tau_1$ this is equal to the sum of the residues to the left of the contour, i.e.

$$\begin{aligned} Q = \mp i\sum_{n=0}^{\infty} \frac{(-n-\frac{1}{2})(n-\frac{3}{2})!}{n!\,(-\frac{1}{2})!} F(-n, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha)\psi_{\frac{1}{2}-n}(\tau)e^{(n-\frac{1}{2})(s_1+i\theta)}+ \\ +\sum_{n=2}^{\infty} \frac{(-n-1)(n-1)!\,(-n-\frac{3}{2})!}{(-\frac{1}{2})!} F(-n-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha)\times \\ \times(-nC_n\psi_n(\tau))(-1)^n e^{n(s_1+i\theta)}. \end{aligned} \tag{16}$$

The latter series, which we call R, is convergent for all τ, θ. Therefore the continuation of Q round a cut from $\tau = \tau_1$, $\theta = 0$ to $\tau = \tau_2$, $\theta = 0$ is $2R-Q$; and continuation a second time round gives again Q.

Also, continuing the series for $\mp(Q-R)$ in (16) into the region $\tau_2 < \tau < \tau_1$, we obtain

$$\begin{aligned} &\mp(Q-R) \\ &= i\sum_{n=0}^{\infty} \frac{(-n-\frac{1}{2})(n-\frac{3}{2})!}{n!\,(-\frac{1}{2})!}\left[\frac{(-\frac{1}{2})!\,n!}{(n-\frac{1}{2})!}\sin\alpha+O(n\cos^{2n}\alpha)\right]\psi_{\frac{1}{2}-n}(\tau)e^{(n-\frac{1}{2})(s_1+i\theta)} \\ &= i\sum_{n=0}^{\infty} \frac{\frac{1}{2}+n}{\frac{1}{2}-n}\sin\alpha\,\psi_{\frac{1}{2}-n}(\tau)e^{(n-\frac{1}{2})(s_1+i\theta)} \end{aligned} \tag{17}$$

plus a function analytic for $\tau > \tau_2$. Expression (17), for $\tau > \tau_1$, is equal (since the residue of $\pi/\cos\pi\nu$ at $\nu = \frac{1}{2}-n$ is $(-1)^{n-1}$) to

$$\mp\left[\frac{\sin\alpha}{2\pi i}\int\limits_{-\infty i}^{\infty i} \frac{\pi}{\cos\pi\nu}\,\frac{\nu-1}{\nu}\psi_\nu(\tau)e^{\nu(\mp i\pi-s_1-i\theta)}\,d\nu+X\right] \tag{18}$$

according as $\theta \lesseqgtr 0$, where

$$X = \sin\alpha\sum_{n=0}^{\infty}\pi(n+1)C_n\psi_n(\tau)e^{n(s_1+i\theta)} \tag{19}$$

(with $C_0 = 1$, $C_1 = 0$ as usual, the term $n = 0$ coming from the pole at $\nu = 0$). X converges everywhere. Hence for $\tau < \tau_1$ expression (17) becomes

$$-i\sin\alpha\sum_{n=1}^{\infty}(-1)^{n-1}\pi\frac{n-\frac{1}{2}}{n+\frac{1}{2}}\psi_{n+\frac{1}{2}}(\tau)(-1)^n e^{-(n+\frac{1}{2})(s_1+i\theta)}\mp X; \tag{20}$$

and so when $\mp(Q-R)$ encircles $\tau = \tau_1$ in the positive direction (considering $(\tau, -\theta)$ as a right-handed system of axes, so that $\theta < 0$ comes first in an encirclement of $(\tau_1, 0)$ in the positive sense starting from $\tau > \tau_1$) it increases by $-2X$. Hence Q 'has a period of $\pm 2X$ at $\tau = \tau_1$' (i.e. increases by this quantity when the point is encircled in the positive sense), according as it has been continued from its starting-point via $\theta \lessgtr 0$.

We also have, for $\tau < \tau_1$, $\theta \lessgtr 0$, with the positive integers the only poles to the right of the contour,

$$P = i\sin\alpha \frac{1}{2\pi i} \int\limits_{-\infty i}^{\infty i} \frac{\pi}{\sin \pi\nu}\left(1-\frac{1}{\nu}\right)\psi_\nu(\tau)e^{\nu(\mp i\pi - s_1 - i\theta)}\,d\nu. \tag{21}$$

When $\tau > \tau_1$ this is the sum of the residues at the poles to the left of the contour; these are all double. Hence,

$$\begin{aligned} P &= i\sin\alpha \sum_{n=0}^{\infty} (-1)^n \frac{d}{d\nu}\left[(\nu+n)\left(1-\frac{1}{\nu}\right)\psi_\nu(\tau)e^{\nu(\mp i\pi - s_1 - i\theta)}\right]_{\nu=-n} \\ &= i\sin\alpha \sum_{n=0}^{\infty} \frac{d}{d\nu}\left[(\nu+n)\left(1-\frac{1}{\nu}\right)\psi_\nu(\tau)e^{-\nu(s_1+i\theta)}\right]_{\nu=-n} + \\ &\qquad\qquad + i\sin\alpha \sum_{n=0}^{\infty} (\mp i\pi)\left(1+\frac{1}{n}\right)(-nC_n\psi_n(\tau))e^{n(s_1+i\theta)}, \end{aligned} \tag{22}$$

and the second term is $\mp X$. Hence, when P encircles $\tau = \tau_1$ in the positive direction starting from $\tau < \tau_1$ (so that $\theta > 0$ comes first) it increases by $2X$.

Thus, if continued via $\theta > 0$ (right-hand sheet), $P+Q$ has zero period at $\tau = \tau_1$, $\theta = 0$: in fact there is no singularity there at all since, by (21) and (18), $P+Q$ differs by a function regular at $\tau = \tau_1$, $\theta = 0$ from

$$\frac{\sin\alpha}{2\pi i} \int\limits_{-\infty i}^{\infty i} \left[\frac{\pi}{\cos \pi\nu} + i\frac{\pi}{\sin \pi\nu}\right] \frac{\nu-1}{\nu} \psi_\nu(\tau)e^{\nu(i\pi - s_1 - i\theta)}\,d\nu, \tag{23}$$

which converges uniformly in the neighbourhood of $(\tau_1, 0)$ since the term in square brackets is $2\pi i e^{-i\pi\nu}\operatorname{cosec} 2\pi\nu$. Thus there is no singularity of $P+Q$ (except B) on the right-hand sheet: on the left-hand sheet, however, there is a point S where $P+Q$ has period $4X$. If a cut is made from this point to infinity $P+Q$ is one-valued on the cut Riemann surface, taking values as shown in Fig. 2, in the different regions. As $\psi = \Im(P+Q)$ encircles the point S (which corresponds of course to the point at infinity in the physical plane) it increases by $\Im(4X)$. By (19) this is not zero for

all τ, θ: hence the stream-function is many-valued in the physical plane and the method (when $f_n(\tau_1)$ is taken as e^{-ns_1}) is useless in this example. This is due to the presence of circulation.

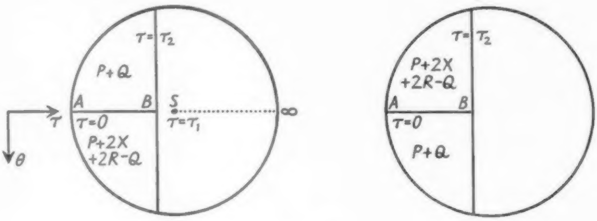

FIG. 2.

4. The correct value of $f_n(\tau_1)$

To overcome this difficulty I thought of choosing a different $f_n(\tau_1)$ more symmetrical to $\psi_n(\tau)$. The reasons for this are far clearer from the method of ref. (**2**) which I evolved subsequently. $\psi_{-n}(\tau_1)$ was first tried: this got rid of part of the period $\mathfrak{I}(4X)$ and pointed clearly the way to the correct value, whose *raison d'être* is demonstrated far more generally as well as convincingly in ref. **2**,

$$f_n(\tau_1) = \frac{\psi_{-n}(\tau_1)+2\tau_1\psi'_{-n}(\tau_1)}{1-n}, \tag{24}$$

where ψ'_{-n} denotes the derivative of ψ_{-n}. The work is here displayed with this latter value used.

Consider, for $\theta \lessgtr 0$, the integral

$$\frac{1}{2\pi i}\int\limits_{-\infty i}^{\infty i} \frac{(\nu-1)(-\nu-1)!\,(\nu-\frac{3}{2})!}{(-\frac{1}{2})!} F(\nu-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha)\psi_\nu(\tau)\times$$
$$\times\frac{\psi_{-\nu}(\tau_1)+2\tau_1\psi'_{-\nu}(\tau_1)}{1-\nu}\, e^{i\nu(\mp\pi-\theta)}\, d\nu. \tag{25}$$

For $\tau < \tau_2$, this is equal to the sum of the residues of the integrand at its poles on the right (these are double poles at the non-negative integers), i.e. to

$$-\sum_{n=0}^{\infty}(-1)^n\frac{d}{d\nu}\left[(\nu-n)^2\frac{(\nu-1)(-\nu-1)!\,(\nu-\frac{3}{2})!}{(-\frac{1}{2})!}F(\nu-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha)\psi_\nu(\tau)\times\right.$$
$$\left.\times\frac{\psi_{-\nu}(\tau_1)+2\tau_1\psi'_{-\nu}(\tau_1)}{1-\nu}\, e^{-i\nu\theta}\right]_{\nu=n}-$$
$$-\sum_{n=0}^{\infty}(\mp i\pi)(-1)^n\frac{(-1)^{n-1}(n-1)(n-\frac{3}{2})!}{n!\,(-\frac{1}{2})!}F(n-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha)\psi_n(\tau)nC_n\times$$
$$\times\frac{\psi_n(\tau)+2\tau_1\psi_n'(\tau_1)}{1-n}\, e^{-in\theta}, \tag{26}$$

which we write as $Q \pm T$, where T converges everywhere and $T \to 0$ as the Mach number at infinity tends to zero. Also, as this happens, the integral (25) tends to the low-speed integral (8), since $\psi_\nu(\tau) \sim \tau^{\frac{1}{2}\nu}$,

$$\frac{\psi_{-\nu}(\tau_1)+2\tau_1\psi'_{-\nu}(\tau_1)}{1-\nu} \sim \frac{\tau_1^{-\frac{1}{2}\nu}+2\tau_1(-\frac{1}{2}\nu\tau_1^{-\frac{1}{2}\nu-1})}{1-\nu} = \tau_1^{-\frac{1}{2}\nu}, \tag{27}$$

and $(\tau/\tau_1)^{\frac{1}{2}\nu}e^{-i\nu\theta} = [(q^2/q_G^2)/(1/q_G^2)]^{\frac{1}{2}\nu}e^{-i\nu\theta} = (qe^{-i\theta})^\nu = \zeta^\nu$. But (8) is the low-speed Q. Hence the Q here defined tends to the low-speed Q as the Mach number at infinity tends to zero and so is a reasonable extension thereof.

When $\tau > \tau_1$, the integral (25) is equal to the sum of its residues at $\frac{1}{2}-n$ $(n \geqslant 0)$ and $-n$ $(n \geqslant 2)$, i.e. to

$$\sum_{n=0}^{\infty} \frac{(\mp i)(-n-\frac{1}{2})(n-\frac{3}{2})!}{n!\,(-\frac{1}{2})!} F(-n,-\tfrac{1}{2};\tfrac{1}{2};\sin^2\alpha)\psi_{\frac{1}{2}-n}(\tau)\times$$
$$\times\frac{\psi_{n-\frac{1}{2}}(\tau_1)+2\tau_1\psi'_{n-\frac{1}{2}}(\tau_1)}{n+\frac{1}{2}}e^{i(n-\frac{1}{2})\theta}+$$
$$+\sum_{n=2}^{\infty}\frac{(-n-1)(n-1)!\,(-n-\frac{3}{2})!}{(-\frac{1}{2})!}F(-n-\tfrac{1}{2},-\tfrac{1}{2};\tfrac{1}{2};\sin^2\alpha)(-nC_n\psi_n(\tau))\times$$
$$\times\frac{\psi_n(\tau_1)+2\tau_1\psi'_n(\tau_1)}{n+1}(-1)^n e^{in\theta}, \tag{28}$$

which we write as $\mp W+R$, where R converges everywhere. Thus the continuation of Q for $\tau > \tau_1$ via $\theta \lessgtr 0$ is $R\mp(T+W)$, and its continuation right round a cut from $\tau = \tau_1$, $\theta = 0$ to $\tau = \tau_2$, $\theta = 0$ is $2R-Q$; and continuation a second time round gives again Q.

Now W can be rewritten as

$$i\sum_{n=0}^{\infty}\frac{(-n-\frac{1}{2})(n-\frac{3}{2})!}{n!\,(-\frac{1}{2})!}\left[\frac{(-\frac{1}{2})!\,n!}{(n-\frac{1}{2})!}\sin\alpha+O(n\cos^{2n}\alpha)\right]\psi_{\frac{1}{2}-n}(\tau)\times$$
$$\times\frac{\psi_{n-\frac{1}{2}}(\tau_1)+2\tau_1\psi'_{n-\frac{1}{2}}(\tau_1)}{n+\frac{1}{2}}e^{i(n-\frac{1}{2})\theta}$$
$$= i\sin\alpha\sum_{n=0}^{\infty}(\tfrac{1}{2}-n)^{-1}\psi_{\frac{1}{2}-n}(\tau)\big(\psi_{n-\frac{1}{2}}(\tau_1)+2\tau_1\psi'_{n-\frac{1}{2}}(\tau_1)\big)e^{i(n-\frac{1}{2})\theta} \tag{29}$$

plus a function analytic for $\tau > \tau_2$. This series is

$$\mp\left[-\frac{\sin\alpha}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\pi}{\cos\pi\nu}\frac{\psi_\nu(\tau)}{\nu}(\psi_{-\nu}(\tau_1)+2\tau_1\psi'_{-\nu}(\tau_1))e^{\nu(\mp i\pi-i\theta)}\,d\nu+X\right] \tag{30}$$

according as $\theta \lessgtr 0$, where

$$X = \pi\sin\alpha\sum_{n=0}^{\infty}C_n\psi_n(\tau)(\psi_n(\tau_1)+2\tau_1\psi'_n(\tau_1))e^{in\theta}, \tag{31}$$

which converges everywhere. For $\tau<\tau_1$, therefore, (29) becomes

$$\pm\sin\alpha\Big[\sum_{n=1}^{\infty}(n+\tfrac{1}{2})^{-1}\psi_{n+\frac{1}{2}}(\tau)\big(\psi_{-n-\frac{1}{2}}(\tau_1)+2\tau_1\psi'_{-n-\frac{1}{2}}(\tau_1)\big)(\mp i)e^{-(n+\frac{1}{2})i\theta}-$$

$$-\sum_{n=0}^{\infty}\pi\psi_n(\tau)C_n\big(\psi_n(\tau_1)+2\tau_1\psi_n'(\tau_1)\big)e^{-in\theta}\Big]\mp X = Z\mp Y,\text{ say,} \tag{32}$$

where $$Y=\pi\sin\alpha\sum_{n=0}^{\infty}C_n\psi_n(\tau)\big(\psi_n(\tau_1)+2\tau_1\psi_n'(\tau_1)\big)(e^{in\theta}+e^{-in\theta}), \tag{33}$$

which is *purely real*. Thus W, when continued round $\tau=\tau_1$, $\theta=0$ in the positive direction ($\theta<0$ first), decreases by $2Y$: hence the period of $Q=R\mp(T+W)$ is $\pm 2Y$ according as it has been continued from its starting-point via $\theta\lessgtr 0$.

Consider next the integral

$$i\sin\alpha\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\pi}{\sin\pi\nu}\Big(1-\frac{1}{\nu}\Big)\psi_\nu(\tau)\frac{\psi_{-\nu}(\tau_1)+2\tau_1\psi'_{-\nu}(\tau_1)}{1-\nu}e^{\nu(\mp i\pi-i\theta)}\,d\nu, \tag{34}$$

which when $\tau<\tau_1$ is equal to

$$-i\sin\alpha\sum_{n=0}^{\infty}\frac{d}{d\nu}\Big[(\nu-n)\Big(1-\frac{1}{\nu}\Big)\psi_\nu(\tau)\frac{\psi_{-\nu}(\tau_1)+2\tau_1\psi'_{-\nu}(\tau_1)}{1-\nu}e^{-i\nu\theta}\Big]_{\nu=n}-$$

$$-i\sin\alpha\sum_{n=0}^{\infty}(\mp i\pi)\Big(1-\frac{1}{n}\Big)\psi_n(\tau)\,nC_n\frac{\psi_n(\tau_1)+2\tau_1\psi_n'(\tau_1)}{1-n}e^{-in\theta}=P\pm(Y-X), \tag{35}$$

with X, Y as in (31), (33). The P here defined reduces to the low-speed P as the Mach number at infinity tends to zero, by arguments similar to those for Q. When $\tau>\tau_1$ the integral (34) is

$$i\sin\alpha\sum_{n=0}^{\infty}\frac{d}{d\nu}\Big[(\nu+n)\Big(1-\frac{1}{\nu}\Big)\psi_\nu(\tau)\frac{\psi_{-\nu}(\tau_1)+2\tau_1\psi'_{-\nu}(\tau_1)}{1-\nu}e^{-i\nu\theta}\Big]_{\nu=n}+$$

$$+i\sin\alpha\sum_{n=0}^{\infty}(\mp i\pi)\Big(1+\frac{1}{n}\Big)\big(-nC_n\psi_n(\tau)\big)\frac{\psi_n(\tau_1)+2\tau_1\psi_n'(\tau_1)}{1+n}e^{in\theta}=H\mp X,\text{ say.} \tag{36}$$

Hence P continues for $\tau>\tau_1$ into $H\mp Y$, so that in encircling $\tau=\tau_1$, $\theta=0$ in the positive direction ($\theta>0$ first) it increases by $2Y$. As in § 3, therefore, $P+Q$ is one-valued on the Riemann surface cut to prevent encirclement of S, is regular except at S and B, and has period $4Y$ at S. ψ, however, being the imaginary part of $P+Q$, has no period at S since Y is real, and so is one-valued on the uncut Riemann surface. The solution for ψ is therefore quite possible physically; and an exact flow of

a compressible fluid past a contour with circulation has been mathematically specified.

The series for ψ in different parts of the Riemann surface are indicated in Fig. 3. The value when $\tau_2 < \tau < \tau_1$ is obtainable by splitting W as

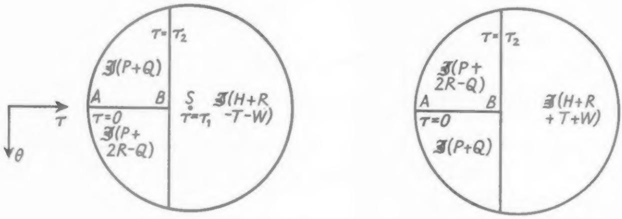

FIG. 3.

in (29) into a term continuable round $\tau = \tau_1$ and one convergent for $\tau > \tau_2$, and using $\psi = \mathfrak{I}(P+R\mp T\mp W)$.

Strictly, an investigation into the one-valuedness of x, y as well as that of ψ is necessary, since x, y may have *constant* periods though ψ has none. Further work is omitted here, however, since the required investigation is given in ref. (**2**). It is clear that the method given there is greatly preferable to the present one, as well for the reasons given in § 1 as for the fact that it gives a single series for ψ convergent throughout $\tau < \tau_s$, which will be more useful than those of the present paper especially around $\tau = \tau_1$ and τ_2 (series of the type of (47) of Part I will also be convenient in this region): this single series is also such that it does not involve relations in the low-speed hodograph plane, which our § 2 shows to be rather involved in even a simple example.

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THE METHOD OF CHARACTERISTICS FOR PROBLEMS OF COMPRESSIBLE FLOW INVOLVING TWO INDEPENDENT VARIABLES

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PART II. INTEGRATION ALONG A MACH LINE. THE RADIAL FOCUSING EFFECT IN AXIALLY SYMMETRICAL FLOW

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SUMMARY

The paper deals with the growth and decay of disturbances along Mach lines in isentropic, irrotational, steady, two-dimensional or axially symmetrical, supersonic flow; in particular, the distribution of disturbances is investigated along a Mach line in axially symmetrical flow on which the velocity is constant. As an example, the field of flow in the entry of a contractor of circular cross-section is calculated from the focusing laws, and the analytical expressions are compared with the results of the numerical methods of Massau and Tupper.

The disturbance generated by the diffuser entry leads to a singularity of the flow pattern on the axis, the nature of which is investigated within the framework of linear theory.

I. Integration along a Mach line

THE physical significance of the characteristics lies in their role as carriers of small disturbances. The whole pattern of a shock-free flow, for example, in a duct with uniform initial flow, can be interpreted as the cumulative effect of small disturbances generated by the walls, and the idea therefore suggests itself to study their growth or decay along the characteristics.†

The key to this investigation is found in the characteristic equation (11 a) of Part I,‡ as far as the Mach lines are concerned. If we employ general orthogonal coordinates based on one of the families of Mach lines, this equation is an identity, and hence the derivative with respect to β of the left-hand side also vanishes. The new equation thus arrived at governs the distribution along the Mach lines of the normal derivatives of the velocity components and the variables of state.

If we restrict ourselves, for the sake of simplicity, to the case of an isentropic, irrotational, steady flow of a perfect gas, the new equation may

† Cf. also ref. (1), § II, 11.

‡ Ref. (2). Since many results of the §§ II, III, IV, and VI of Part I are used in the present report, the notation of Part I is followed throughout, and the equations are numbered through so that all numbers below (40) refer to the formulae of Part I.

be written in the form

$$\frac{d\kappa_\beta}{h_\alpha d\alpha}+\kappa_\beta^2+A(\alpha)\kappa_\beta+B(\alpha) = 0 \tag{41}$$

(the details of the calculation are given in the appendix (i)).

The normal derivatives of the velocity components are related to the curvature of the normals by

$$\begin{aligned} \partial v_\beta/h_\beta\,\partial\beta &= F(\alpha)-\frac{\gamma-1}{\gamma+1}v_\alpha\kappa_\beta, \\ \partial v_\alpha/h_\beta\,\partial\beta &= v_\beta\kappa_\beta-\frac{\gamma+1}{\gamma-1}(v_\beta/v_\alpha)F(\alpha). \end{aligned} \tag{42}$$

(41) can also be written in the form

$$\frac{d^2h_\beta}{h_\alpha^2\,d\alpha^2}+\left(A(\alpha)-\frac{\partial h_\alpha}{h_\alpha^2\,\partial\alpha}\right)\frac{dh_\beta}{h_\alpha d\alpha}+B(\alpha)h_\beta = 0, \tag{43}$$

and this equation governs the divergence or convergence of the Mach lines.

The functions A, B, F depend only on the state of motion on, and the shape of, the Mach lines along which the equations hold; (41) and (42) therefore permit us to calculate the 'propagation' along a Mach line of a discontinuity of the normal derivatives of the velocity components as soon as the shape of, and the state of motion on, this Mach line are known. Unfortunately, A and B are of a very complicated form (cf. appendix (i)), and the equations (41) to (43) have so far been investigated only for Mach lines on which the state of motion is constant and which are therefore straight lines. In two-dimensional flow such a Mach line must belong to a region of uniform flow or to a simple wave, and an investigation of the focusing effect on it cannot contribute much to the known facts about these flow patterns. But for axially symmetrical flow we are led to a theory of the radial focusing effect which is the fundamental feature which distinguishes it from plane, shock-free, supersonic flow.

II. The focusing of disturbances in axially symmetrical flow

II.1. On a Mach line along which the state of motion is constant we have

$$\partial v_\beta/\partial\alpha = \partial v_\alpha/\partial\alpha = \kappa_\alpha = 0, \tag{44}$$

and if we wish to include the axis in the region under investigation, we must also have

$$R = v_r/r \equiv 0 \tag{44a}$$

(v_r denotes the radial velocity component). Equation (41) then reduces to

$$\frac{d}{d\alpha}(h_\beta\kappa_\beta)+\frac{\sin\epsilon}{2r}h_\alpha h_\beta\kappa_\beta = 0 \tag{45}$$

(cf. appendix (ii)), and since $h_\alpha\, d\alpha \sin\epsilon = dr$ we have

$$h_\beta \kappa_\beta = mr^{-\frac{1}{2}}, \tag{46}$$

where m is a constant. By virtue of (3), a second integration gives

$$h_\beta = \frac{c+r^{\frac{1}{2}}}{1+c}, \quad \text{with} \quad 1+c = \sin\epsilon/2m, \tag{47}$$

where the constant of integration has been chosen so that $h_\beta = 1$ when $r = 1$. Finally,

$$\kappa_\beta = \frac{\sin\epsilon}{2} r^{-\frac{1}{2}}(c+r^{\frac{1}{2}})^{-1} \quad (r \neq 0), \tag{48}$$

and since $F(\alpha) = 0$, owing to (44), equation (42) becomes

$$\frac{\partial v_\alpha}{\partial\beta} = -\frac{\gamma+1}{\gamma-1}\frac{v_\beta}{v_\alpha}\frac{\partial v_\beta}{\partial\beta} = v_\beta h_\beta \kappa_\beta. \tag{49}$$

The rates of change of the velocity components per unit length in the direction normal to the Mach line are given by $\partial v_\alpha/h_\beta \partial\beta$ and $\partial v_\beta/h_\beta \partial\beta$, respectively. Both are proportional to

$$r^{-\frac{1}{2}}(c+r^{\frac{1}{2}})^{-1}.$$

The changes of the velocity components along the normal, from the Mach line to a neighbouring one, are given by $(\partial v_\alpha/\partial\beta)\, d\beta$ and $(\partial v_\beta/\partial\beta)\, d\beta$ respectively, and they are proportional to $r^{-\frac{1}{2}}$.

II.2. Equations (47) and (48) give h_β and κ_β all along the Mach line as soon as either is known at any one point of it. If κ_β, the curvature of the normal, is discontinuous at some point of the Mach line, then κ_β is discontinuous at every point of it, and (48) determines the variation of the magnitude of this discontinuity.

The curvature of the streamlines is discontinuous, too, where they cross a Mach line with such a disturbance. By virtue of Bernoulli's equation we can write (12c) in the form

$$\kappa_s = -\frac{1}{w}\frac{\partial w}{h_n \partial n},$$

replacing α and β by s and n when they refer to a coordinate system based on the streamlines. But stream and Mach coordinates are related by

$$\pm h_\beta\, d\beta = h_s\, ds \sin\mu \pm h_n\, dn \cos\mu,$$

where the upper sign corresponds, as always, to the upper sign in (11), and hence, by (44),

$$\kappa_s = -\frac{\cos\mu}{w}\frac{\partial w}{h_\beta \partial\beta}.$$

By (21), (11), (10), (44) and the isentropic relation

$$\rho \propto a^{2/(\gamma-1)}, \tag{50}$$

this becomes

$$\kappa_s = \mp \{2/(\gamma+1)\}\cos^2\mu \sin\mu\, \kappa_\beta. \tag{51}$$

Equations (51) and (47) allow us to draw some conclusions as regards the maximum disturbance which a wall may be permitted to introduce into a uniform, axially symmetrical flow without producing a shock-wave. Let us consider a straight pipe of radius $r = 1$, with uniform flow upstream of a certain point $x = 0$, where its cross-section begins to change in such a way that the radius of curvature of the meridian suddenly assumes a finite value R_{s1}. The discontinuity of the streamline curvature persists along the 'leading' Mach line through the point $x = 0$, $r = 1$ which separates the uniform flow upstream from the non-uniform flow downstream. It is the last Mach line on which the conditions (44) hold. On it, $\theta = 0$ and, by (16), $\epsilon = \mp\mu$ and hence, by (49) and (48),

$$R_s = 1/\kappa_s = N(c+r^{\frac{1}{2}})r^{\frac{1}{2}},$$

where
$$N = (\gamma+1)/\sin^2\mu\cos^2\mu = \frac{(\gamma+1)M_0^4}{M_0^2-1} > 0$$

and M_0 denotes the Mach number of the uniform flow upstream. Thus,

$$1+c = R_{s1}/N,$$

and by (47),
$$h_\beta = 1-N(1-r^{\frac{1}{2}})/R_{s1}. \tag{52}$$

If R_{s1} is negative, that is, if the disturbance is an expansion, h_β is positive all along the leading Mach line inside the duct, but if it is a compression, $h_\beta > 0$ only when $r^{\frac{1}{2}} > 1-R_{s1}/N$. When $h_\beta = 0$ the leading Mach line meets an envelope of its family, and the presence of an envelope indicates the occurrence of a shock-wave. The inequality

$$R_{s1} > (\gamma+1)M_0^4/(M_0^2-1) \tag{53}$$

is therefore a necessary condition for shock-free flow in the entry of a supersonic contractor of circular cross-section.† In the case of a given contractor, with small initial curvature, (53) determines two limiting Mach numbers between which shock-free flow is possible. The Mach number which admits the highest initial curvature of a contractor is $M_0 = \sqrt{2}$; and if the initial radius of curvature is smaller than $4(\gamma+1)$ times the radius of the pipe, a shock-free flow is not possible for any Mach number.

A similar condition can be deduced for two-dimensional flow. Instead of (46) we have $h_\beta\kappa_\beta = B = \text{const.}$, and if t denotes the length on the leading Mach line measured from the wall, and $h_\beta = 1$ at the wall,

$$h_\beta = Bt+1,$$
$$R_s = \mp\tfrac{1}{2}N(t+B^{-1})\sin\mu.$$

† This condition has also been deduced by K. Friedrichs.

For a symmetrical channel of initial width $2b$,

$$h_\beta = 1 \mp \tfrac{1}{2}N(b-y)/R_{s1} \tag{54}$$

on the leading Mach lines, if y denotes the distance from the axis. The condition for shock-free flow becomes

$$R_{s1} > \tfrac{1}{2}(\gamma+1)M_0^4 b/(M_0^2-1). \tag{55}$$

The Mach number which admits the highest initial curvature of a two-dimensional contractor is again, $M_0 = \sqrt{2}$; and if the initial radius of curvature is smaller than $(\gamma+1)$ times the initial width of the symmetrical channel, a shock-free flow is not possible for any Mach number.

Equations (53) and (55) are, of course, necessary but not sufficient conditions for shock-free flow.

A curious case occurs when $R_{s1} = N$ so that h_β vanishes just on the axis.

II.3. The question now arises whether similar results can be derived for disturbances of higher orders, that is, discontinuities of the curvatures, and normal velocity derivatives, of higher orders. Let us consider a Mach line on which

$$\left.\begin{aligned} \partial v_\beta/\partial\alpha &= \ldots = \partial^{m+1}v_\beta/\partial\beta^m\partial\alpha = 0,\\ \partial v_\alpha/\partial\alpha &= \ldots = \partial^{m+1}v_\alpha/\partial\beta^m\partial\alpha = 0,\\ \kappa_\alpha &= \ldots = \partial^m\kappa_\alpha/\partial\beta^m = R = \ldots = \partial^m R/\partial\beta^m = 0,\\ \kappa_\beta &= \ldots = \partial^{m-1}\kappa_\beta/\partial\beta^{m-1} = 0. \end{aligned}\right\} \tag{56}$$

Without loss of generality, we can choose the parameter, α, so that $h_\alpha \equiv 1$ on this Mach line. If we now differentiate (11 a) $(m+1)$ times with respect to β and make use of (56), (11), (10), (4), and (50), we find that

$$\partial^m(h_\beta\kappa_\beta)/\partial\beta^m, \quad \partial^{m+1}v_\beta/\partial\beta^{m+1}, \quad \partial^{m+1}v_\alpha/\partial\beta^{m+1} \quad \text{vary as} \quad r^{-\frac{1}{2}} \tag{57}$$

on the Mach line. If $m = 0$, h_β is given by (47), but if $m \geqslant 1$,

$$\begin{gathered} h_\beta = 1, \quad \partial h_\beta/\partial\beta = \ldots = \partial^{m-1}h_\beta/\partial\beta^{m-1} = 0,\\ \partial^m h_\beta/\partial\beta^m \propto (r^{\frac{1}{2}}-1) \end{gathered} \tag{58}$$

(the constant of integration is chosen so that $\partial^m h_\beta/\partial\beta^m = 0$ when $r = 1$).

The equations (46) to (48) and (57), (58) are a generalization of a known result of linear theory (ref. 3). If we differentiate the equation for the linearized potential,

$$\phi_{rr} + \frac{1}{r}\phi_r - \alpha^2\phi_{xx} = 0, \qquad \alpha^2 = M_0^2 - 1,$$

(the suffixes denote partial derivatives) with respect to x and eliminate

ϕ_{xrr} by means of the identities

$$d\phi_{xx} = \phi_{xxx}\,dx + \phi_{xxr}\,dr,$$
$$d\phi_{xr} = \phi_{xxr}\,dx + \phi_{xrr}\,dr$$

we arrive at

$$(dx^2 - \alpha^2\,dr^2)\phi_{xxx} + d\phi_{xr}\,dr - d\phi_{xx}\,dx + \frac{1}{r}\phi_{xr}\,dr^2 = 0. \tag{59}$$

But on a Mach line where the state of motion is constant, we have

$$dx \pm \alpha\,dr = 0,$$
$$d\phi_x = \phi_{xx}\,dx + \phi_{xr}\,dr = 0,$$

and (59) reduces to $$\alpha\left(\frac{1}{r} + 2\frac{d}{dr}\right)\phi_{xx} = 0,$$

that is, $$\phi_{xx} \propto r^{-\frac{1}{2}}, \tag{60}$$

provided $M_0 \neq 1$. A similar result can obviously be established for the derivatives of any order.

In a certain sense there is full agreement between the linear and the non-linear theory. If we interpret ϕ_{xx} as the rate of change in axial velocity from one Mach line to another, the radial focusing law is the same in both theories; and this law does not depend on the order of the disturbance, provided all those of lower order vanish. The main improvement which the non-linear theory brings about is the distinction between the rate of change from one Mach line to another and the rate of change per unit length. It is an unexpected result that the error to which the linear treatment may lead if these two notions are confused increases in importance, not as we approach the axis, but as we go away from it.

If we consider the entry of a circular duct with shock-free flow our results may be summed up as follows. If we calculate the velocity components on a Mach line which belongs to the same family as the leading Mach line we obtain the same result from both theories, provided we take the 'second' Mach line near enough to the leading one for the velocity components to be given by the linear terms in their Taylor series with respect to β.† This is true for any 'smoothness' of the wall. But the simple Mach line pattern of the linear theory is distorted in the general theory, and the distortion depends on the smoothness of the wall.

III. The entry of a supersonic contractor of circular cross-section

III.1. The theory of radial focusing enables us to explain the surprising results obtained by Tupper in his paper on a supersonic contractor (ref. 4)

† It will be seen from the results of § IV that this assertion cannot be extended to a certain portion of the second Mach line near the axis.

to which this investigation owes a good deal of stimulation. Tupper's work was based on the linearized equations of motion, and as far as the linear theory goes the explanation of the flow pattern in the entry of an axially symmetrical duct will be completed by the results of § IV.

In order to investigate how far the focusing laws given in § II.1 can explain the flow pattern which the non-linear theory predicts for the entry of such a duct, a characteristic very near to the entry of Tupper's contractor was recomputed, using the full, non-linear equations of motion as described in § VI and appendix (ii) of Part I (ref. **2**). The result is shown in Table 1. The initial Mach number ($M_0 = 1{\cdot}9$) and the shape

TABLE 1

Second Characteristic of Contractor

r_2	x_2	w^2	a^2	θ	v_x	v_r	(70)	(71)
·99985	·07000	2·5019	·69962	−·00437	1·5817	−·00691	−·006878	−·007115
·89900	·23023	2·5012	·69977	−·00460	1·5815	−·00727(5)	−·007244	−·007512
·79811(5)	·39036(5)	2·5003	·69994	−·00488	1·5812	−·00772	−·007675(5)	−·007983
·69718	·55041	2·4992	·70016	−·00521	1·5809	−·00824	−·008195	−·008556
·59617	·71033(5)	2·4979	·70042	562	1·5804(5)	−·00888	−·008839	−·009273
·49508	·87013	2·4962	·70076	615	1·5799	−·00972	−·009665	−·010206
·39387	1·02973	2·4939	·70123	686	1·5792	−·01083	−·010780	−·011491(5)
·29245	1·18906	2·4904	·70192	791	1·5780	−·01248	−·012407	−·013428
·19083	1·34793	2·4845	·70310	−·00967	1·5762	−·01524	−·015108	−·016850
·13985	1·42713	2·4793	·70414	−·01117	1·5745	−·0176	−·01735	−·019954
·08868	1·50602	2·4702	·70596	−·0137	1·5715	−·0215	−·02094	−·025795
·05781	1·55309	2·4595	·70811	−·0165	1·5680	−·0259	−·02417	−·033343
·02851	1·59705	2·4361	·71278	−·0216	1·5604	−·0337	−·02207	−·054686
·01489	1·61701	2·4046	·71909	−·0260	1·5501	−·0403	+·10043	−·109100
·00752	1·62755	2·3524	·72952	−·0258	1·5332	−·0396		
0	1·63829	2·1865	·76270	0	1·4787	0		

Wall: $1 - r = 0{\cdot}0312740x^2 - 0{\cdot}00064855x^3$.

Undisturbed stream: $M_0 = 1{\cdot}9$, $w_0^2 = 2{\cdot}5156794$, $a_0^2 = 0{\cdot}6968641$, $u_0 = w_0 = 1{\cdot}5860893$.

of the wall, (62), are the same as in Tupper's paper, $\gamma = 1{\cdot}4$ and the 'second' Mach line was chosen to meet the wall at $x = 0{\cdot}07$ so that it starts very nearly, at least, at the same place as the characteristic $m = 2$ in Tupper's case. The table gives the coordinates, x_2 and r_2, of the points which were computed together with the values, at those points, of the squares of the velocity magnitude, w, and the speed of sound, a, of the angle, θ, between the velocity direction and the axis, and of the radial and axial velocity components, v_r and v_x. All velocities are understood as multiples of the critical speed of sound, a_s.

According to the linear theory, the second Mach line is straight and parallel to the leading one. The present computation shows that it is, in fact, nearly straight; but the distance between the two Mach lines decreases

appreciably as the radius decreases. The leading Mach line meets the axis at $x = \sqrt{(M_0^2-1)} = 1{\cdot}61555$, and the table therefore shows that this distance (measured in the axial direction) falls from 0·0700 at the wall to 0·0227 at the axis. Since it is the radial velocity component which shows the most unexpected behaviour, according to Tupper's graphs (ref. **3**), it has once more been plotted in Fig. 1 according to the results

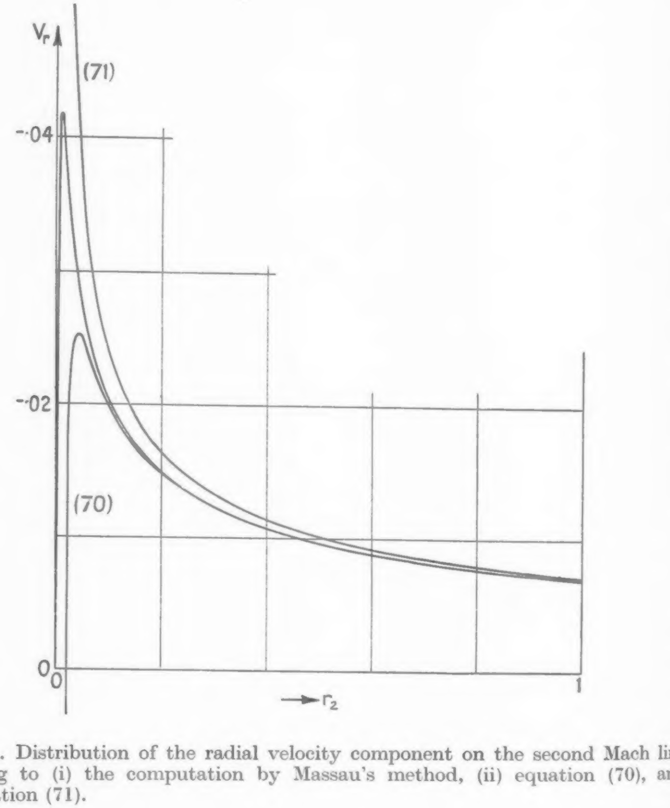

FIG. 1. Distribution of the radial velocity component on the second Mach line according to (i) the computation by Massau's method, (ii) equation (70), and (iii) equation (71).

of the present computation. A comparison shows that the rapid variation and the peak in the distribution of v_r near the axis are even more noticeable than in Tupper's results.

We now proceed to compare the results of our computation with those obtained for the radial focusing effect. If we choose our system of general orthogonal coordinates so that $\beta = 0$ on the leading Mach line, the

equation of the second Mach line will be

$$\beta = \text{const.} = \beta_2 \ll 1,$$

and the radial velocity component on it is given by the Taylor series

$$v_{r2} = \beta_2(\partial v_r/\partial\beta)_{\beta=0} + \tfrac{1}{2}\beta_2^2(\partial^2 v_r/\partial\beta^2)_{\beta=0} + \dots, \tag{61}$$

since v_r vanishes on the leading Mach line. The variation of $\partial v_r/\partial\beta$ on the leading Mach line is determined by the focusing law for first-order disturbances, and the equations (49) and (46) show that

$$(\partial v_r/\partial\beta)_{\beta=0} \propto r^{-\frac{1}{2}}.$$

Clearly, a part only of the distribution of v_r shown in Fig. 1 can be approximated by the first term of the series (61). However, we must not forget that equations (46) to (49) only describe the focusing of the first-order disturbances, whereas the wall, the shape of which is given by the equation

$$1-r = \tfrac{1}{32}x^2 - 0{\cdot}00450448x^4 + 0{\cdot}000216430x^6 \tag{62}$$

(cf. ref. **4**), introduces also discontinuities of the third- and fifth-order curvatures of the streamlines at the point $x = 0$.

The calculation of the focusing laws for these higher order curvatures, in a case where the lower order ones do not vanish, would be a tedious task, and we shall try to avoid it. The description of the first step of the Massau process given in § VI.1 of Part I (ref. **2**) shows that only the following data about the wall enter into the computation of the second characteristic: the values of x, r, and θ at the points where the leading and second Mach line meet the wall. With our choice of the (x, r)-coordinate system, and with an equation of the form

$$1-r = Ax^2 + Bx^3 + Cx^4 + Dx^5 + \dots \tag{63}$$

for the shape of the wall (so that $r = 1$, $\theta = 0$ at $x = 0$) we can start the second characteristic with any proposed values of x, r, θ by an appropriate choice of two of the coefficients, while all the others are put equal to zero. For our purpose, a wall of the shape (62) is equivalent to one of the shape

$$1-r = 0{\cdot}0312740x^2 - 0{\cdot}00064855x^3. \tag{64}$$

In fact, even if a contractor was designed in such a way as to avoid any discontinuity of the first-order curvature of the wall at the entry, we should still be justified in computing its second characteristic as if the shape of the wall were of the form (63) with $C = D = \dots = 0$.

III.2. We therefore propose to evaluate the first two terms of the series (61) and to compare the result with that of the computation. The radial velocity is

$$v_r = v_\alpha \sin\epsilon + v_\beta \cos\epsilon$$

and hence
$$\frac{\partial v_r}{\partial\beta} = \frac{\partial v_\alpha}{\partial\beta}\sin\epsilon+\frac{\partial v_\beta}{\partial\beta}\cos\epsilon+(v_\alpha\cos\epsilon-v_\beta\sin\epsilon)\frac{\partial\epsilon}{\partial\beta}$$
$$= \frac{2}{\gamma+1}\,wh_\beta\kappa_\beta\cos^2\mu$$
on the leading Mach line, by virtue of (49), (4), (16), and since $v_\alpha = w\cos\mu$. Taking account of (46) and (47) we therefore have
$$(\partial v_r/\partial\beta)_{\beta=0} = \frac{w\cos^2\mu\sin\epsilon}{(\gamma+1)(1+c)}r_1^{-\frac{1}{2}}, \tag{65}$$
where the suffix 1 indicates that we mean the radius of the point on the leading Mach line which lies on the same normal as the point on the second Mach line at which we propose to calculate v_r.

In order to find the focusing law for the second-order disturbance we have to differentiate (11 a) twice with respect to β taking account of (44). We do not now assume, as we did in § II.3, that the first-order disturbance vanishes (i.e. κ_β, etc. are now $\neq 0$), and a somewhat lengthy transformation of the same type as that given in the appendix (i) is needed in order to deduce an equation for the variation of $\partial(h_\beta\kappa_\beta)/\partial\beta$ with α along the leading Mach line.† It is found to have the form
$$\frac{d}{h_\alpha d\alpha}\frac{\partial(h_\beta\kappa_\beta)}{\partial\beta}+\frac{\sin\epsilon}{2r}\frac{\partial(h_\beta\kappa_\beta)}{\partial\beta}+\frac{\sin\epsilon}{4r}\left(3\tan\epsilon+\frac{3\gamma-1}{\gamma+1}\cot\epsilon\right)(h_\beta\kappa_\beta)^2+$$
$$+\frac{\sin^2\epsilon}{8r^2}\left[\tan\epsilon+\left(4\frac{\gamma-1}{\gamma+1}-11\right)\cot\epsilon\right]h_\beta^2\kappa_\beta = 0. \tag{66}$$
In contrast to (45), this equation is inhomogeneous, which means that a first-order disturbance at the entry of the duct always generates a second-order disturbance even if the equation of the wall, as, for example, (62), shows no discontinuity of the second-order curvature of the wall. The algebra which leads to (66) makes it appear highly probable that the equations for the variation of the disturbances of any order will be found to be inhomogeneous whenever a disturbance of lower order is present. We must expect a disturbance of any order to generate disturbances of all higher orders.

If we introduce (46) and (47) into (66) and note that $dr = h_\alpha d\alpha\sin\epsilon$ on the leading Mach line, we can write (66) in the form
$$\frac{d}{dr}\left[\sqrt{r}\frac{\partial}{\partial\beta}(h_\beta\kappa_\beta)\right] = m^2(A_1r^{-2}-B_1r^{-\frac{3}{2}}),$$
with $A_1 = \frac{c}{4}\left(\frac{7\gamma+15}{\gamma+1}\cot\epsilon-\tan\epsilon\right)$, $\quad B_1 = \tan\epsilon-\frac{\gamma+4}{\gamma+1}\cot\epsilon$.

† This algebra has been omitted through lack of space. A detailed account of it will be found in the appendix of A.R.C. 10649.

This integrates to

$$\frac{\partial}{\partial\beta}(h_\beta \kappa_\beta) = m^2(2B_1 r^{-1} - A_1 r^{-\frac{3}{2}} + \delta r^{-\frac{1}{2}}). \tag{67}$$

In order to find $\partial h_\beta/\partial\beta$ itself we note that, by (3) and (44),

$$h_\beta \frac{\partial \kappa_\beta}{\partial\beta} = \frac{h_\beta}{h_\alpha}\frac{\partial}{\partial\beta}(h_\alpha \kappa_\beta) = \frac{h_\beta}{h_\alpha}\frac{\partial}{\partial\beta}\left(\frac{\partial h_\beta}{h_\beta\,\partial\alpha}\right) = \frac{h_\beta}{h_\alpha}\frac{\partial^2}{\partial\alpha\partial\beta}\log h_\beta,$$

and therefore, again by virtue of (3),

$$\frac{\partial}{\partial\beta}(h_\beta \kappa_\beta) = \frac{\partial h_\beta}{\partial\beta}\frac{\partial h_\beta}{h_\beta h_\alpha \partial\alpha} + \frac{h_\beta}{h_\alpha}\frac{\partial^2}{\partial\alpha\partial\beta}\log h_\beta = \frac{\partial}{h_\alpha \partial\alpha}\left(\frac{\partial h_\beta}{\partial\beta}\right).$$

It follows that

$$\frac{\partial h_\beta}{\partial\beta} = \frac{1}{\sin\epsilon}\int\limits_1^r \frac{\partial}{\partial\beta}(h_\beta \kappa_\beta)\,dr$$

$$= \frac{2m^2}{\sin\epsilon}[A_1(r^{-\frac{1}{2}}-1) + B_1 \log_e r - \delta(1-r^{\frac{1}{2}})], \tag{68}$$

if the lower limit is chosen so that $\partial h_\beta/\partial\beta = 0$ when $r = 1$.

From (67) it can be deduced (cf. appendix (iii)) that

$$\left(\frac{\partial^2 v_r}{\partial\beta^2}\right)_{\beta=0} = \frac{2m^2}{\gamma+1} w \cos^2\epsilon[B_2 r^{-1} - A_2 r^{-\frac{3}{2}} + \delta r^{-\frac{1}{2}}], \tag{69}$$

with $\quad A_2 = \frac{c}{4}(3\tan\epsilon + 11\cot\epsilon), \qquad B_2 = -\tan\epsilon - \frac{3\gamma+7}{\gamma+1}\cot\epsilon,$

and combining this with (65) and (61) we have

$$v_{r2} = \frac{w\cos^2\epsilon\sin\epsilon}{4(\gamma+1)(1+c)^2}\beta_2[\{4(1+c)+\beta_2\delta\sin\epsilon\}r_1^{-\frac{1}{2}} +$$
$$+\beta_2 r_1^{-1}\sin\epsilon(B_2 - A_2 r_1^{-\frac{1}{2}})] + O(\beta_2^3), \tag{70}$$

provided $r_1 \neq 0$ (since $\theta = 0$, $\mu = \mp\epsilon$ on the leading Mach line; $m = \frac{1}{2}\sin\epsilon/(1+c)$). The constants of integration, c and δ, are found by comparing (65) and (69) with (64), at $r_1 = 1$.† β_2 can then be evaluated, by the help of (47) and (68), from the distance between the leading and the second Mach line, measured along the normal through the point of intersection of the second Mach line and the wall. The values of the constants which correspond to the initial data of the computation are thus found to be

$$c = 0{\cdot}3341401, \quad \delta = -0{\cdot}9108727, \quad \beta_2 = 0{\cdot}0371515.$$

In order to compare the distribution of v_r on the second Mach line which (70) predicts, with that obtained from the computation, the values of r_1 which correspond to the point x_2, r_2 given in the table were computed with

† Cf. last footnote.

due regard to the slope and curvature of the normals; v_{r2} could then be evaluated from (70) as a function of r_2, and the result is found in the last column but one of Table 1. For further comparison, v_{r2} was evaluated from the equation

$$v_{r2} = \frac{w\cos^2\epsilon\sin\epsilon}{(\gamma+1)(1+c)}\beta_2 r_1^{-\frac{1}{2}}, \tag{71}$$

which is obtained from (61) and (65) if account is taken only of the first-order disturbance at the entry of the contractor, and this result has been entered into the last column of the table.

The distributions of v_r as found from (70) and (71) are also plotted in Fig. 1, and it is seen that the rough approximation obtained from (71) is much improved by taking account of the focusing effect for the second-order disturbance. The most striking feature of the curve (70) is that it shows indeed a peak, owing to the higher negative powers of r_1 in the 'small' terms of the formula.

III.3. Our comparison shows that over a considerable part of the entry of an axially symmetrical duct, the field of flow can be approximated very satisfactorily by analytical expressions derived from the focusing laws (46), (47) and (67), (68) for the first- and second-order disturbances. However, as the radius decreases, a systematic discrepancy appears, and the agreement becomes unsatisfactory for small values of r.

Before we accept this result it may be useful to check the accuracy of the numerical method. In any computation small random errors are introduced at every step, and the question arises whether they tend to cancel out in our case. In order to investigate this question we shall imagine that a small error is introduced into the value of one of the variables at some point of the second characteristic, and that the computation is then continued once on the basis of the correct values at that point, and a second time on the basis of the erroneous values, assuming in both cases that no further error occurs. Instead of comparing these two computations we can compare the second characteristics of two contractors which have initial radii roughly equal to the radius at which the error was introduced, and suitably chosen small differences in the initial Mach number, the initial radius, and the shape of the wall. We now apply the equation (71). There will be small differences in the values of the constants on its right-hand side, and it follows that the absolute difference in the values of v_{r2} increases as $r_1^{-\frac{1}{2}}$ while the relative difference is independent of the radius.†

† The argument can be followed up in more detail, including, for example, the modification of the position of the second Mach line which is caused by the introduction of the error. Cf. Appendix B of A.R.C. 10649.

The argument shows that small computational errors are, in fact, focused like disturbances, and a total error which is considerably bigger than the sum of the small random errors introduced at every step of the computation must therefore accumulate from them. A systematic discrepancy must exist between the computed distribution of the radial velocity on the second Mach line and the true distribution. However, the focusing of errors can cause a breakdown of the Massau method only in the cases where infinitesimal disturbances are actually focused to finite strength.† In our case we cannot expect the errors to increase at a higher rate than the radial velocity itself, and since the random errors incurred in the present computation are of the order of 0·1 per cent. of the respective values of v_r, at most, their focusing cannot account for the discrepancy between the curves of Fig. 1.

The improvement of the approximation which was brought about by the use of equation (70) instead of (71) suggests that the range over which the field of flow is represented satisfactorily by analytical expressions derived from the focusing laws can be extended farther by taking account of the third and higher order disturbances which are generated by those of the first and second order. However, our analytical method for the calculation of the second characteristic cannot be extended right to the axis. The normal which meets the leading Mach line on the axis constitutes a natural limit for the use of Taylor series of the type (61) for the calculation of the dependent variables. Comparison of (71) and (70) shows how the convergence of the series deteriorates as this limit is approached. In fact, all our formulae indicate a singularity in the field of flow at the point where the leading Mach line meets the axis, and the dependent variables are probably not analytic functions of β at all over any finite range, however small, on the normal through this point.

It appears desirable, therefore, to investigate the singularity directly. A first step in this direction is described in the following section.

IV. The singularity on the axis in linear theory

IV.1. If the condition (53) is satisfied, we shall not expect that a discontinuity of the velocity components themselves is caused by the focusing effect on the leading Mach line, and the computation provides no contrary evidence. Moreover, it has been shown in § II.3 that the agreement between the predictions of the linear and the non-linear theory improves as the axis is approached along the leading Mach line, and we shall therefore try to find a first approximation for the field of flow near the singularity by the help of the linear theory.

† Similar conclusions appear to have been arrived at also by Tollmien (ref. **5**).

The linearized potential which describes the flow pattern predicted by linear theory must be a solution of the equation

$$\phi_{rr}+\frac{1}{r}\phi_r-\alpha^2\phi_{xx}=0, \qquad \alpha^2=M_0^2-1, \tag{72}$$

which vanishes upstream of, and on, the leading Mach line, and the second derivatives of which vary like $r^{-\frac{1}{2}}$ on the downstream side of this line and the first radial derivative of which vanishes all along the axis. The expression

$$\phi=\frac{2}{3\pi}\int\limits_0^\pi (x'+\alpha r\cos u)^{\frac{3}{2}}\delta\,du, \tag{73}$$

with $x'=x-\alpha$, $\delta=1$ when $x'+\alpha r\cos u\geqslant 0$, and $\delta=0$ otherwise, satisfies these requirements. That it is a solution of (72) is known (ref. **3**). The conditions for ϕ and ϕ_r are obviously satisfied, and the behaviour of (73) on the downstream side of the leading Mach line is found by introducing $q=(x'+\alpha r)/2\alpha r$ and substituting $\sin^2(u/2)=q\sin^2\theta$, whence

$$\phi_x=\frac{1}{\pi}\int\limits_0^\pi (x'+\alpha r\cos u)^{\frac{1}{2}}\delta\,du$$

$$=\frac{2q}{\pi}(2\alpha r)^{\frac{1}{2}}\int\limits_0^{\pi/2}\cos^2\theta(1-q\sin^2\theta)^{-\frac{1}{2}}\delta\,d\theta\to 0 \quad\text{as}\quad q\to 0,$$

$$\phi_{xx}=\frac{1}{2\pi}\int\limits_0^\pi (x'+\alpha r\cos u)^{-\frac{1}{2}}\delta\,du$$

$$=\frac{1}{\pi}(2\alpha r)^{-\frac{1}{2}}\int\limits_0^{\pi/2}(1-q\sin^2\theta)^{-\frac{1}{2}}\delta\,d\theta\to(8\alpha r)^{-\frac{1}{2}} \quad\text{as}\quad q\to 0.$$

This transformation also shows that the velocity components vary like $r^{\frac{1}{2}}$, and their derivatives like $r^{-\frac{1}{2}}$, on the lines where x'/r is constant, that is, on the straight lines through the singular point $x'=0$, $r=0$. On the axis, downstream of this point,

$$\phi_x=\sqrt{x'}.\dagger \tag{74}$$

In order to find the behaviour of the solution on the 'reflected' Mach line which slopes downstream from the singular point, we substitute $p=(\alpha r-x')/2\alpha r$ and $\sigma=(\pi-u)/2$, whence

$$\phi\to\frac{8}{9\pi}(2\alpha r)^{\frac{3}{2}}, \qquad \phi_x\to\frac{2}{\pi}(2\alpha r)^{\frac{1}{2}}, \qquad \phi_{xx}\to\infty \quad\text{as}\quad p\to\pm 0. \tag{75}$$

† An indication of this result is to be found in ref. (**3**).

The behaviour of (73) near the axis, the reflected and the leading Mach lines, is best found by writing the potential as the product of $r^{\frac{3}{2}}$ and a function of $1/p$, p, and q, respectively, and substituting in (72). This leads to a hypergeometric differential equation for each of the functions; in particular, if we write

$$\phi = \frac{4}{3\pi}(2\alpha r)^{\frac{3}{2}}K(p),$$

(72) gives

$$p(1-p)K''+(2p-1)K'-\tfrac{9}{4}K = 0.$$

Near $p = 0$, the general solution is given by a series of the form

$$K = a_0+a_1 p+a_2 p^2\log p+O(p^2);\dagger$$

a_0 and a_1 are found from (75). Near the reflected Mach line the velocity derivatives therefore vary like

$$r^{-\frac{1}{2}}\log p.\ddagger$$

The disturbance on the leading Mach line is reflected from the axis not as a disturbance but as a logarithmic singularity.§

From (75) and the corresponding value of ϕ_r, the series for ϕ valid near the reflected Mach line is found to be

$$\phi = \frac{8}{9\pi}(2\alpha r)^{\frac{3}{2}}\big(1-\tfrac{9}{4}p-\tfrac{9}{32}p^2\log p+O(p^2)\big), \tag{76}$$

and similarly, the series near the leading Mach line and near the axis are found to be

$$\left.\begin{aligned} \phi &= \tfrac{1}{4}(2\alpha r)^{\frac{3}{2}}F(\tfrac{1}{2},\tfrac{1}{2};3;q), \\ \phi &= \tfrac{2}{3}(x'-\alpha r)^{\frac{3}{2}}F(\tfrac{1}{2},-\tfrac{3}{2};1;1/p), \end{aligned}\right\} \tag{77}$$

where F denotes the hypergeometric series.

IV.2. In order to check whether the linearized potential, (73), really provides an approximation for the general, non-linear field of flow near the singularity, we must insert the expressions (76), (77), in the non-linear differential equation,

$$(a^2-\Phi_x^2)\Phi_{xx}-2\Phi_x\Phi_r\Phi_{xr}+(a^2-\Phi_r^2)\Phi_{rr}+\frac{a^2}{r}\Phi_r = 0,$$

$$a^2 = a_0^2+\frac{\gamma-1}{2}(w_0^2-\Phi_x^2-\Phi_r^2),$$

† Cf. ref. (**6**), § 8.

‡ This result has been confirmed, and extended, by Ward (ref. **7**).

§ With reference to Fig. 1, it may be noted that (76) does not imply a cusp of the velocity distribution along the second Mach line; its point of intersection with the reflected Mach line lies between the axis and the point where v_r is a maximum.

for the velocity potential $\Phi = w_0(x+\phi)$. It is seen that the non-linear terms are, in fact, small compared with the linear terms when r is small, provided p is not small. But, however small we may choose r, there is always a range of values of p for which this is not true, owing to the logarithmic singularity of the second derivatives of ϕ.

Moreover, if one tries to construct a closer approximation for the non-linear potential in the form of an expansion in powers of r with coefficients depending on p, with the first term given by (73), one finds that the expansion fails to converge if p is sufficiently small. The methods employed in this investigation do not permit us to draw any conclusion as regards the flow pattern in the immediate neighbourhood of the reflected Mach line and downstream of it.

V. Conclusions

It is shown that the growth and decay of disturbances along the Mach lines is governed by an ordinary differential equation of the first order (eqn. (**41**)). The convergence and divergence of the Mach lines is governed by an ordinary, homogeneous, second-order differential equation (eqn. (43)).

These equations are integrated for the case of a straight Mach line in axially symmetrical flow which occurs, for example, at the entry of a circular duct with uniform flow upstream. The discontinuities of the normal velocity derivatives of any order vary like $r^{-\frac{1}{2}}$ along such a Mach line, provided that no discontinuities of the derivatives of lower order are present. This result is compared with the focusing law of linear theory. On the other hand, a disturbance of any order must be expected to generate disturbances of all higher orders.

A disturbance of the first order is equivalent to a discontinuity of the curvature of the streamlines where they cross the Mach line in question. It follows from the focusing law that the initial curvature of the wall of a supersonic diffuser must be limited if the flow is to be shock-free (eqn. (53)). For a contractor with given initial curvature of the wall there is only a certain range of Mach numbers for which a shock-wave can be avoided. (A similar condition applies to the initial radius of curvature of a two-dimensional contractor (eqn. 55).)

As an example, in § III, the entry of the contractor is calculated which was investigated by Tupper (ref. **4**) with the help of a numerical method based on linear theory. The field of flow is computed by the Massau method, as described in Part I (ref. **2**), and it is shown that a fair approximation can be obtained for the greatest part of the contractor entry by using analytical expressions which take account only of the

disturbances of the first and second order. Moreover, it is seen that the numerical method suffers from the disadvantage that small computational errors are also focused like disturbances. On the other hand, the field of application of the analytical method is limited, near the axis, by the normal which meets the leading Mach line on the axis; the convergence deteriorates as this limit is approached.

The focusing law indicates a singularity of the field of flow at the point where the axis is met by the leading Mach line, which carries the disturbance. It is shown that, on linear theory, the field of flow is represented by the linearized potential (73), which is evaluated in terms of hypergeometric functions. The velocity components are seen to vary like $r^{\frac{1}{2}}$, and their derivatives like $r^{-\frac{1}{2}}$, on all straight lines through the singular point. The discontinuity of the velocity derivatives on the leading Mach line is reflected from the axis as a logarithmic singularity. If the velocity components are plotted along any line which crosses the leading Mach line and its reflection these curves suffer a sudden change of slope at the former place and they have a vertical tangent (but no cusp) at the latter.

Upstream of the reflected Mach line this solution provides an approximation for the flow pattern of the general, non-linear theory near the singular point. However, the method fails to provide any information about the field of flow in the neighbourhood of the reflected Mach line.

I should like to thank Professor Goldstein very sincerely for his encouragement and helpful comment.

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APPENDIX

(i) From (50) and (11),
$$\frac{d\rho}{\rho} = \frac{2}{\gamma-1}\frac{dv_\beta}{v_\beta}$$
and eliminating ρ from the equation of continuity, (10), we have
$$\frac{\partial v_\alpha}{h_\alpha \partial\alpha}+\frac{2}{\gamma-1}\frac{v_\alpha}{v_\beta}\frac{\partial v_\beta}{h_\alpha \partial\alpha}+\lambda\frac{\partial v_\beta}{h_\beta \partial\beta}+v_\alpha\kappa_\beta+v_\beta\kappa_\alpha+R = 0, \tag{78}$$
with
$$\lambda = (\gamma+1)/(\gamma-1).$$
For a perfect gas, Bernoulli's equation can be written
$$\frac{2}{\gamma-1}a^2+w^2 = v_\alpha^2+\lambda v_\beta^2 = \text{constant}$$

(by virtue of (11)); and hence

$$v_\alpha\,dv_\alpha = -\lambda v_\beta\,dv_\beta, \tag{79}$$

and (78) becomes

$$\partial v_\beta/\partial\beta = h_\beta\,F(\alpha)-(v_\alpha/\lambda)h_\beta\kappa_\beta, \tag{80}$$

with

$$F(\alpha) = \left(\frac{v_\beta}{v_\alpha}-\frac{2}{\gamma+1}\frac{v_\alpha}{v_\beta}\right)\frac{\partial v_\beta}{h_\alpha\,\partial\alpha}-\frac{1}{\lambda}(v_\beta\kappa_\alpha+R); \tag{81}$$

whence (42) follows. Moreover, Bernoulli's equation and (11) and (79) may be used to eliminate p, ρ, a, and v_α from (11 a), which thus assumes the form

$$-\frac{\lambda}{v_\alpha}\left\{v_\beta^2+\left(\frac{2}{\lambda}-1\right)v_\alpha^2\right\}\frac{\partial v_\beta}{h_\alpha\,\partial\alpha}+w^2\kappa_\alpha+v_\beta\,R = 0. \tag{82}$$

If we now differentiate (82) with respect to β, (80) with respect to α, and eliminate $\partial^2 v_\beta/\partial\alpha\partial\beta$ and also eliminate $\partial h_\alpha/\partial\beta$ by (3) and $\partial\kappa_\alpha/\partial\beta$ by (5), we arrive at the following equation:

$$\frac{4}{\gamma+1}\left(\frac{\partial\kappa_\beta}{h_\alpha\,\partial\alpha}+\kappa_\beta^2\right)+\left(\frac{v_\alpha\kappa_\beta}{\lambda v_\beta}-\frac{F}{v_\beta}\right)\left[\frac{2v_\beta}{v_\alpha}R+\left(Q+\frac{2w^2}{(\gamma-1)v_\alpha^2}\right)\kappa_\alpha-\frac{1}{v_\alpha}\left(Q+\lambda^2\frac{v_\beta^2\,w^2}{v_\alpha^4}\right)\frac{\partial v_\beta}{h_\alpha\,\partial\alpha}\right]+$$
$$+\frac{Q\kappa_\beta}{v_\alpha}\left(F+\frac{v_\beta}{v_\alpha}\frac{\partial v_\beta}{h_\alpha\,\partial\alpha}\right)-\frac{v_\beta}{v_\alpha^2}\frac{\partial R}{h_\beta\,\partial\beta}+\frac{w^2}{v_\alpha^2}\kappa_\alpha^2+\frac{Q}{v_\alpha}\left(\frac{\partial F}{h_\alpha\,\partial\alpha}-\kappa_\alpha\frac{\partial v_\beta}{h_\alpha\,\partial\alpha}\right) = 0, \tag{83}$$

with

$$Q = 2+\lambda(v_\beta^2-v_\alpha^2)/v_\alpha^2.$$

Now

$$R = \frac{1}{r}(v_\alpha\sin\epsilon+v_\beta\cos\epsilon),$$

and

$$\partial r/h_\beta\,\partial\beta = \cos\epsilon,$$

and hence, by virtue of (79), (80), and (4),

$$\frac{\partial R}{\partial\beta} = \frac{2}{\gamma+1}\frac{v_\alpha\cos\epsilon}{r}\kappa_\beta+\frac{F}{rv_\alpha}(v_\alpha\cos\epsilon-\lambda v_\beta\sin\epsilon)-\frac{R}{r}\cos\epsilon, \tag{84}$$

which shows that (83) is, in fact, of the form (41). Equation (43) follows from (41) if we note that, by (3),

$$\frac{\partial^2 h_\beta}{\partial\alpha^2} = h_\alpha^2 h_\beta\left(\frac{\partial\kappa_\beta}{h_\alpha\,\partial\alpha}+\kappa_\beta^2\right)+h_\beta\kappa_\beta\frac{\partial h_\alpha}{\partial\alpha}.$$

(ii) Equation (45) is best obtained by taking account of (44) when differentiating (82) with respect to β. Moreover, it follows from (44) that $F(\alpha)\equiv 0$, and hence that (49) holds on the leading Mach line and that

$$\partial^2 v_\beta/\partial\alpha\partial\beta = -(v_\alpha/\lambda)\partial(h_\beta\kappa_\beta)/\partial\alpha. \tag{85}$$

In order to eliminate $\partial\kappa_\alpha/\partial\beta$ we note that, by (5), (44), and (3),

$$\partial\kappa_\alpha/\partial\beta = -\partial(h_\beta\kappa_\beta)/h_\alpha\,\partial\alpha; \tag{86}$$

finally, $\partial R/\partial\beta$ can be eliminated by the help of (84) and (44) and we are left with

$$2v_\alpha\,\partial(h_\beta\kappa_\beta)/\partial\alpha-h_\alpha\,v_\beta\frac{\cos\epsilon}{r}h_\beta\kappa_\beta = 0,$$

from which (45) follows, since $R = 0$ implies

$$v_\beta/v_\alpha = -\tan\epsilon \tag{87}$$

on the leading Mach line.

(iii) It also follows from (87) that

$$\frac{\partial^2 v_r}{\partial\beta^2} = \frac{\partial^2 v_\alpha}{\partial\beta^2}\sin\epsilon+\frac{\partial^2 v_\beta}{\partial\beta^2}\cos\epsilon+2\frac{\partial\epsilon}{\partial\beta}\left(\frac{\partial v_\alpha}{\partial\beta}\cos\epsilon-\frac{\partial v_\beta}{\partial\beta}\sin\epsilon\right)+(v_\alpha\cos\epsilon-v_\beta\sin\epsilon)\frac{\partial^2\epsilon}{\partial\beta^2} \tag{88}$$

on the leading Mach line. By differentiating (81) with respect to β and taking

account of (85), (49), (86), (84), (45), and repeatedly of (44) and (87), one finds that

$$\frac{\partial F}{\partial \beta} = -\frac{v_\alpha \cos \epsilon}{(\gamma+1)\lambda r} h_\beta \kappa_\beta,$$

and hence, from (80) and since $F = 0$, that

$$\frac{\partial^2 v_\beta}{\partial \beta^2} = -\frac{v_\alpha}{\lambda}\frac{\partial(h_\beta \kappa_\beta)}{\partial \beta} - \frac{v_\beta}{\lambda}(h_\beta \kappa_\beta)^2 - \frac{v_\alpha \cos \epsilon}{(\gamma+1)\lambda r} h_\beta^2 \kappa_\beta$$

on the leading Mach line. From (79) and (87),

$$\partial^2 v_\alpha / \partial \beta^2 = \lambda \frac{\partial}{\partial \beta}\left(\tan \epsilon \frac{\partial v_\beta}{\partial \beta}\right),$$

and if account is also taken of (4) and (49), (88) assumes the form

$$\frac{\partial^2 v_r}{\partial \beta^2} = \frac{2v_\alpha}{\gamma+1}\cos \epsilon \frac{\partial(h_\beta \kappa_\beta)}{\partial \beta} - \frac{4v_\alpha}{\gamma+1}\sin \epsilon (h_\beta \kappa_\beta)^2 + \frac{\cos \epsilon}{\gamma+1}\left(v_\beta \sin \epsilon - \frac{v_\alpha}{\lambda}\cos \epsilon\right)\frac{h_\beta^2 \kappa_\beta}{r},$$

whence (69) follows by virtue of (87), (67), (46), and (47).

ASSESSMENT OF ERRORS IN APPROXIMATE SOLUTIONS OF DIFFERENTIAL EQUATIONS

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SUMMARY

The term assessment is applied to any process which enables us to set rigid bounds to the error or to estimate its value. It is shown that upper and lower bounds can be assigned whenever the Green's function of the problem is one-signed; this is true in many important problems. Another method is applicable to step-by-step solutions of ordinary differential equations, linear or non-linear, and depends on using the 'index' of the process of integration. Lastly, the error in a linear problem can be estimated when an approximation to the Green's function is known.

1. Introduction: Meaning of assessment

THE term *error* is used to signify the difference between the value of an unknown as given by some process of approximation and its true value. When the differential equation is written as an expression equated to zero, the value of this expression for the approximate solution is called the *residual*. The word *assessment* is used to cover any process which enables us to estimate or delimit. There are two principal kinds of assessments of error:

(*a*) the fixing of rigid upper and lower bounds to the error;

(*b*) the estimation, more or less closely, of the error.

When it is possible to fix rigid bounds to the error these are usually rather widely separated and the assessment is correspondingly crude, but when the bounds are both rigid and close we have a most useful assessment. However, when we have not discovered how to set rigid bounds, or when these are insufficiently close, we must have recourse to some method of estimation. Such an estimation may yield results of ample accuracy for the purposes of applied mathematics even when lacking ideal precision. It is to be remarked that we exclude strict *evaluation* of the error from consideration since, if this could be done, we should have a method of exact solution. An essential requirement is that any method of assessment shall be applicable throughout the whole region of integration.

We shall here confine attention to differential equations with boundary conditions which render the solution unique. No attempt at a general treatment will be made, but the following items will be discussed:

1. Bounds to the errors of ordinary and partial differential problems having one-signed Green's functions.

2. Estimation of the errors of the step-by-step solutions of ordinary differential equations, or sets of these, with one-point boundary conditions.
3. Estimation of the errors of linear problems when an approximation to the Green's function is known.

Item 1 is concerned only with linear problems and may appear of very limited applicability; in fact it covers some very important problems such as Poisson's equation with fixed boundary values of the unknown. The basis of the assessment is the value of the residual, and it is assumed that this can be found from the approximate solution. For approximations such as those provided by the methods of Rayleigh–Ritz, of Galerkin, or of the thickness parameter the residual will be given as an explicit function, which is particularly convenient. Item 2 covers non-linear problems.

The intention of the writer is to give a concise account of some useful methods and no attempt is made to enter into the minutiae of numerical work.

2. Bounds to the errors in problems having one-signed Green's functions

2.1. *Outline of the method*

We consider an ordinary or partial linear differential equation with boundary conditions of the linear and homogeneous type which render the solution unique. We suppose further that the Green's function of the problem is one-signed.

Let the differential equation be

$$\Delta\phi = f, \tag{2.1,1}$$

where Δ is a linear ordinary or partial differential operator, ϕ is the unknown, and f is a given function of the independent variable or variables. Also let

$$\Delta\sigma = 1 \tag{2.1,2}$$

with the same boundary conditions as for ϕ. We shall call σ the *basic solution*, and it is one-signed on account of our assumption about the Green's function. Suppose that ϕ_a is some approximation to ϕ *which exactly satisfies the boundary conditions*, and let

$$\Delta\phi_a - f = \epsilon, \tag{2.1,3}$$

so that ϵ is the *residual* corresponding to ϕ_a and is, in general, a function of the independent variables. Let

$$\epsilon_1 = \text{absolute maximum value of } \epsilon \text{ in the region of integration,}$$

and

$$\epsilon_2 = \text{absolute minimum value of } \epsilon \text{ in the region of integration.}$$

Then $(\epsilon-\epsilon_1)$ is everywhere negative or zero and $(\epsilon-\epsilon_2)$ everywhere positive or zero in the region.

For definiteness let the Green's function be everywhere *positive* in the region. Then by equations (2.1,1)–(2.1,3)

$$\Delta(\phi-\phi_a+\epsilon_1\sigma) = \epsilon_1-\epsilon \geqslant 0,$$

and consequently
$$\phi-\phi_a+\epsilon_1\sigma \geqslant 0,$$
or
$$\phi \geqslant \phi_a-\epsilon_1\sigma.$$
Also
$$\Delta(\phi-\phi_a+\epsilon_2\sigma) = \epsilon_2-\epsilon \leqslant 0$$
and
$$\phi \leqslant \phi_a-\epsilon_2\sigma.$$
Finally,
$$\phi_a-\epsilon_1\sigma \leqslant \phi \leqslant \phi_a-\epsilon_2\sigma. \tag{2.1,4}$$

If the Green's function were everywhere *negative* we should have

$$\phi_a-\epsilon_2\sigma \leqslant \phi \leqslant \phi_a-\epsilon_1\sigma. \tag{2.1,5}$$

These inequalities call for two remarks:

(*a*) Provided that ϵ_1 and ϵ_2 are small, a rough approximation to σ will usually give sufficient information about the errors in ϕ_a.

(*b*) A large but highly localized error in ϕ_a, or its derivatives, will yield numerically large values for one or both of ϵ_1, ϵ_2 and will greatly widen the bounds in the inequalities. It is therefore important to avoid such large local errors when the present method of assessment is used.

2.2. *Errors in the basic solution*

Let σ_a be an approximation to σ and

$$\Delta\sigma_a-1 = \eta. \tag{2.2,1}$$

Also let η_1, η_2 be the absolute maximum and absolute minimum values, respectively, of η. Then

$$\Delta(\sigma-\sigma_a+\eta_1\sigma) = \eta_1-\eta \geqslant 0$$

and
$$\Delta(\sigma-\sigma_a+\eta_2\sigma) = \eta_2-\eta \leqslant 0.$$

Hence, if the Green's function is *positive*,

$$\frac{\sigma_a}{1+\eta_1} \leqslant \sigma \leqslant \frac{\sigma_a}{1+\eta_2}, \tag{2.2,2}$$

but if the Green's function is *negative*,

$$\frac{\sigma_a}{1+\eta_2} \leqslant \sigma \leqslant \frac{\sigma_a}{1+\eta_1}. \tag{2.2,3}$$

2.3. *Cases where the boundary conditions are not homogeneous*

When the boundary conditions are linear but not homogeneous we can reduce the problem to one with homogeneous conditions as follows. Let β be a convenient function which satisfies the boundary conditions. Then

$$\psi = \phi-\beta \tag{2.3,1}$$

satisfies linear and homogeneous boundary conditions and

$$\Delta\psi = f-\Delta\beta, \tag{2.3,2}$$

where the function on the right-hand side of the equation is known.

2.4. *An important case where the Green's function is one-signed*

By way of example we shall show that the Green's function for Poisson's equation is negative, the region of integration being the space interior to a closed surface B upon which the solution is to vanish.

Let the differential equation to be solved be

$$\nabla^2\phi = \psi, \tag{2.4,1}$$

where ψ is everywhere positive within B and ϕ is zero on B. Then either ϕ is everywhere negative within B or it is positive at some point P within B. If it is positive at P it must be also positive everywhere within a closed surface C surrounding P and vanish on C; moreover, C is entirely within B or coincides with it wholly or partly and ψ is therefore everywhere positive within C. Apply Green's theorem to the space within C.

$$\iiint \left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2+\phi\nabla^2\phi\right\} dxdydz = -\iint \phi\frac{\partial\phi}{\partial\nu}\,dS = 0, \tag{2.4,2}$$

where $\partial\phi/\partial\nu$ is the rate of change of ϕ along the inward drawn normal to C and dS is the element of the surface of C. But according to our hypothesis the integrand is everywhere positive within C and the integral cannot vanish. Accordingly ϕ cannot be positive anywhere within B, and, since ψ is an arbitrary positive function, it follows that the Green's function is always negative. The same theorem is true in two dimensions.

This proposition was applied by the writer some years ago to delimit the errors in approximate solutions of special problems in the theory of elasticity (**1**, **2**). In place of the basic function σ the Prandtl torsional stress function Ψ was used; this satisfies the two-dimensional equation:

$$\nabla^2\Psi+2 = 0 \tag{2.4,3}$$

and vanishes on the closed boundary, so

$$\Psi = -2\sigma. \tag{2.4,4}$$

It was shown to be possible to place very close bounds to the errors in approximations to Ψ itself (**1**, **2**) and to a stress function arising in the St. Venant theory of flexure (**2**).

3. A method for estimating the errors in step-by-step solutions of sets of ordinary differential equations

3.1. *Outline of the method*

The method here described appears to have been first proposed by L. F. Richardson (**3, 4**), who gave it as an example of what he called 'the deferred approach to the limit'. A more recent account (**5**) has been given by the present writer and the practical value of the method, which is applicable to non-linear equations, has been demonstrated by a number of examples.

Briefly, the method is based on the idea of extrapolation towards the limit of the step-by-step solution corresponding to a vanishingly small interval. Suppose that the independent variable is t, the range of integration a to b, and n the number of equal intervals used in the step-by-step process. Then, provided that the process is completely regular and the boundary values of the unknowns are all given for a (say), we may assume that the error in the value of the dependent variable x_r can be expanded in the series

$$\epsilon_r(t) = n^{-k}(e_0+e_1 n^{-1}+e_2 n^{-2}+\text{etc.}), \tag{3.1,1}$$

provided also that n is not too small. The number k is characteristic of the particular process used and is called its *index* by the present writer. Now when the interval is sufficiently small we may as a first approximation retain only the first or dominant term in the expansion and write

$$\epsilon_r(t) = e_0 n^{-k}. \tag{3.1,2}$$

Since k will be known it becomes possible to calculate e_0 when the approximate solution has been obtained for two values of n. Thus we shall have

$$x_r(t) = x_{r_1}(t)+e_0 n_1^{-k} = x_{r_2}(t)+e_0 n_2^{-k},$$

where $x_{r_1}(t)$, $x_{r_2}(t)$ are the approximations to $x_r(t)$ with n_1 and n_2 intervals respectively. The last equations yield

$$e_0 = \frac{x_{r_2}(t)-x_{r_1}(t)}{n_1^{-k}-n_2^{-k}} \tag{3.1,3}$$

and a first approximation to the error is obtained. With three values of n we can similarly calculate e_0 and e_1 and obtain a next approximation, as has been shown by examples, and the process could be carried still further if desired.

It may be noted that the index varies from 1 in the original and relatively crude process given by Euler in the eighteenth century to 4 in the process of Runge and Kutta (**6**). The various methods are briefly reviewed in the writer's paper (**5**).

When the method just described is applied an estimate should first be made of the number of steps needed to provide the desired accuracy. Then a first calculation should be made with, say, half this number of steps. A comparison of the results obtained with the two numbers of steps allows the error to be assessed and partially corrected. If need be, a still larger number of intervals must finally be used.

Evidently the errors introduced by rounding-off the results of a computation will reduce the accuracy of the results yielded by the foregoing process. It will therefore be desirable to minimize such errors by carrying the computations some decimal places farther than might appear strictly necessary. The greatest error which can be introduced by rounding-off can easily be ascertained in any given case.

3.2. *Extension of the method to partial differential equations*

The method just described can sometimes be applied to partial differential problems. It is essential that a perfectly regular process be used and that an index should exist. When this is so, the error can be assessed as before when the results for two similar lattices are known. It may even be possible to dispense with this condition of similarity.

4. Estimation of the errors in approximate solutions of linear problems when an approximation to the Green's function is known

We shall suppose that we have an approximation ψ_a to the solution of (2.3,2) which satisfies the linear and homogeneous boundary conditions exactly and gives a residual

$$\epsilon = f - \Delta(\psi_a + \beta). \tag{4,1}$$

Then, if there is an approximation G_a to the Green's function appropriate to the homogeneous boundary conditions, we may derive an approximation to the error of deficiency in ψ_a at any point by forming the corresponding integral of the product ϵG_a throughout the region.

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